On Analysis of Discrete Spatial Fuzzy Sets in 2 and 3 Dimensions

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Doctoral Thesis
Swedish University of Agricultural Sciences
Uppsala 2005

Acta Universitatis Agriculturae Sueciae
2005:112

ISSN 1652-6880
ISBN 91-576-6911-2

Printed in Sweden by Universitetstryckeriet, Uppsala University, Uppsala, 2005

Abstract

Nataša Sladoje. *On Analysis of Discrete Spatial Fuzzy Sets in 2 and 3 Dimensions*. Doctoral Thesis
ISSN 1652-6880, ISBN 91-576-6911-2

The use of fuzzy set theoretical approaches for representing spatial relationships provides an intuitive way of expressing the diffuse localization and limits of image components. Fuzziness can be present in images as a consequence of noise introduced during the imaging process, in which case it should preferably be removed, and as imprecision inherent to the observed objects, in which case it provides important information that should be utilized during the image analysis process.

A general goal for the research presented in this thesis has been to develop shape analysis methods that can be applied to fuzzy segmented images in 2D and 3D. A demand for the developed methods has been to respect the specific nature of a fuzzy representation of the studied shapes and, especially, the consequences of discretization. We have studied representation and reconstruction of a shape by using moments of both its crisp and fuzzy discretization. We show, both theoretically and statistically, that the precision of estimation of moments of a shape is increased if a fuzzy representation of a shape is used, instead of a crisp one. The signature of a shape based on the distance from the shape centroid is studied and two approaches for its calculation for fuzzy shapes are proposed. A comparison of the performance of fuzzy and crisp approaches is carried out through a statistical study, where a higher precision of shape signature estimation is observed for the fuzzy approaches. The measurements of area, perimeter, and compactness, as well as of volume, surface area, and sphericity, are considered, too. New methods are developed for the estimation of perimeter and surface area of a discrete fuzzy shape. It is shown through statistical studies that the precision of all the observed estimates increases if a fuzzy representation is used and that the improvement is more significant at low spatial resolutions. In addition, a defuzzification method based on feature invariance is designed, utilizing the improved estimates of shape characteristics from fuzzy sets to generate crisp discrete shapes with the most similar shape characteristics. This defuzzification method, performed on a fuzzy segmentation, can be seen as an alternative to a crisp segmentation of an image.

The presented results can be applied wherever precise estimates of shape properties are required, especially in conditions of limited spatial resolution. We have showed, either theoretically, or empirically, that membership resolution available can be successfully utilized to overcome a lack of spatial resolution.

Key words: digital image analysis, shape description, fuzzy sets, moments, signature, area, perimeter, volume, surface area, defuzzification, multiple criteria evaluation

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Contents

On Analysis of Discrete Spatial Fuzzy Sets in 2 and 3 Dimensions

Papers included in the thesis

The thesis is based on the following articles.

I Žunić, J., Sladoje, N. (2000). Efficiency of Characterising Ellipses and Ellipsoids by Discrete Moments. *IEEE Trans. on Pattern Analysis and Machine Intelligence* 22(4):407–414.

II Sladoje, N., Lindblad, J. (2005). Estimation of Moments of Digitized Objects with Fuzzy Borders. In *Proc. 13th Intern. Conf. on Image Analysis and Processing, Lecture Notes in Computer Science* 3617:188–195. Springer-Verlag.

III Sladoje, N., Lindblad, J. (2005). Representation and Reconstruction of Disks by Fuzzy Discrete Moments. *Submitted for journal publication.*

IV Chanussot, J., Nyström, I., Sladoje, N. (2005). Shape Signatures of Fuzzy Star-Shaped Sets Based on Distance from the Centroid. *Pattern Recognition Letters* 26(6):735–746 .

V Sladoje, N., Nyström, I., Saha, P. K. (2003). Perimeter and Area Estimations of Digitized Objects with Fuzzy Border. In *Proc. Discrete Geometry for Computer Imagery, Lecture Notes in Computer Science* 2886:368–377. Springer-Verlag.

VI Sladoje, N., Nyström, I., Saha, P. K. (2005). Measurements of Digitized Objects with Fuzzy Borders in 2D and 3D. *Image and Vision Computing* 23:123–132.

VII Sladoje, N., Lindblad, J., Nyström, I. (2005). Defuzzification of Spatial Fuzzy Sets by Feature Invariance. *Manuscript*.

In Papers I and IV, the author took part in discussions about development of methods and participated in their implementation and testing, as well as writing parts of the text. Paper III resulted from the equally shared work between both authors. The author's contribution to Paper VII was in developing the method and writing the main part of the text. The work presented in Papers II, V, and VI was done mainly by the author, with helpful discussions with the co-authors.

The C and C++ implementations were either done as stand alone programs or as modules in the image processing platform IMP, developed at the Centre for Image Analysis originally by Bo Nordin and with substantial contributions by Joakim Lindblad. Implementations in MATLAB are used in Paper IV.

Faculty opponent is Dr Isabelle Bloch, Ecole Nationale Supérieure des Télécommunications, Paris, France.

Preface

Two years after receiving the master degree at the University of Novi Sad (Serbia and Montenegro) in the field of Discrete Mathematics and Programming, with the master thesis entitled *Discrete Objects: Characterization by Discrete Moments*, the author had the great pleasure to get in contact with the researchers at the Centre for Image Analysis (CBA) in Uppsala and to begin her doctoral studies under the supervision of Prof. Gunilla Borgefors and Doc. Ingela Nyström. The suggested project was very appealing. The idea was to start the research on the analysis of digital shapes represented by fuzzy sets. The author's background in shape analysis, strongly supported by the experience and knowledge of the researchers at CBA, were intended to be utilized in order to develop new shape analysis techniques, applicable to fuzzy segmented shapes. That work would be a continuation of the research on combining both geometry and intensity information from digital images in order to obtain more powerful shape descriptors, carried out at CBA for a number of years. In that way, this thesis is not only a continuation of the author's master thesis, but also a continuation of the work performed in the field of binary and grey-level shape analysis, by Nyström, Borgefors, Svensson, Wählby, and Sintorn. This thesis contains the theoretical results on the development of fuzzy shape analysis methods based on moments, signature, and a number of 2D and 3D geometric shape descriptors (area, perimeter, and compactness measure in 2D, and volume, surface area and sphericity in 3D). Research on defuzzification methods, by utilizing developed fuzzy shape characteristics, is also presented.

The author's wish is to continue with research on the descriptors of fuzzy shapes. One direction is to work on further extensions of the existing shape analysis methods to fuzzy shapes or designing new ones, while another challenge is to investigate various possibilities for application of the obtained results.

1 Introduction

Fuzzy sets are introduced by Zadeh (1965). The interest for developing methods for the analysis of shapes represented as fuzzy sets appeared in Rosenfeld (1979) more than twenty-five years ago, and has been more or less active since then. That has resulted in the fuzzy shape analysis tool-box which contains the extensions of almost all classical shape analysis methods to the fuzzy case, but with varying quality. While some of these tools, like mathematical morphology and distance transforms, are rather intensively studied and well developed, some other, like moment-based methods, are hardly even mentioned in the literature. It is also notable that 2D fuzzy shapes are far more studied than fuzzy shapes in 3D. There are still many challenges in fuzzy shape analysis. Our results, among others, show an evident improvement of the quantitative analysis, primarily in terms of precision and accuracy, if fuzzy shape representations are used instead of crisp ones. This motivates addressing the further tasks. In addition, the increasing interest for fuzzy images in different application fields, first of all in medical imaging, has encouraged the research presented in this thesis.

A general goal has been to develop shape analysis methods that can be applied to fuzzy segmented images. To some extent, it has been possible to rely on the approaches well known in crisp shape analysis, but the demand for all of the developed methods has been to respect the specific nature of a fuzzy representation of the studied shapes. Even though the developed methods are intended to be independent of any specific fuzzy segmentation method, we found it interesting (and important) to have in mind certain ways of creating fuzzy images. Fuzzification methods based on partial area and volume coverage have been of main interest, due to their simplicity and common use. We compare precision of estimates of certain properties of continuous shape derived from its discrete representation, in cases when crisp discrete and fuzzy discrete sets, respectively, are used as the representation.

We show that the precision of estimation of moments of a shape (Papers I–III), of a signature based on distance from the shape centroid (Paper IV), and of measurements of area, perimeter and compactness (Paper V), as well as of volume, surface area, and sphericity (Paper VI), increases, if the fuzzy representation of a shape is used. In all observed cases, the improvement is more significant at low spatial resolutions.

In addition, a defuzzification method based on feature invariance is designed (Paper VII), with a motivation to utilize improved estimations of shape characteristics to generate crisp discrete shapes most similar to continuous shapes which are imaged. This defuzzification method, performed on a fuzzy segmentation, can be seen as an alternative to a crisp segmentation of an image.

About this thesis

The first part of this thesis puts a frame around the work presented in the included papers, appearing as the second part of the thesis. The purpose of the frame is, first, to give an introduction to the fuzzy set theory (Section 2) and to put it in the context of digital images (Sections 3 and 4), and, second, to briefly describe the main work on the analysis of fuzzy spatial sets (Sections 5 and 6), presented in more details in Papers I–VII. The frame is closed with a discussion on the presented results and a selection of some of the future challenges to take (Sections 7 and 8).

Important questions, related to this task, are:

- What is a fuzzy set?
- What is a fuzzy segmented image? Is it just a fancy name for a grey-value image?
- How do we get fuzzy segmented images?
- What are the advantages of using them?
- What are the disadvantages?
- How do we relate them to conventional binary images?
- What can we do with fuzzy images?
- What (more) would we like to do with fuzzy images?

2 Background

The mosaic of our story is created of pieces coming from several fields of knowledge: digital images and their creation, the theory of fuzzy sets, and the analysis of shapes are the main ones. Before we started to combine them in the story on analysis of discrete spatial fuzzy sets, we had learnt the essentials from each of the fields. Let us briefly present these essentials.

2.1 Images

Humans rely a lot on visual perception. An impressive amount of our knowledge about the surrounding world is based on images and vision. Data, existing in the space around us, create just some geometric patterns, more or less familiar to our visual system, but when additionally equipped with different colours, textures, movements in certain directions, such data provide much more. We “see” that it is warm outside, since we see the sunlight, we “see” that the wind is blowing, since we see the tree-crowns moving, we “see” that the watermelon is sweet, because we see that it is deep red inside, we “see” that the woman sitting at the table next to ours is sad, since we see tears on her face... Well, sometimes we are wrong, but all those visual impressions are so strong and suggestive that they make us believe that we have felt, heard, tasted, understood, even though we have only seen.

In order to save our visual memories over time, or to share our impressions with others, we make images of the scenes we have seen. That requires capturing reflected light in each point of the scene, as our visual system does, and plotting the result. To make it practically achievable, only a sample of points is observed, instead of a continuum of the observed piece of space. In such a way, the image is discretized; one way to do that is presented in Figure 1. The domain of an image, initially being a subset of two- (2D) or three-dimensional (3D) real space (R^2 or R^3), is mapped onto a discrete subset of points having (most often) integer coordinates (Z^2) forming an integer grid. The continuous mapping (image function) that assigns the value to each image point in accordance with the light reflected in it, is restricted to the grid points, and each grid point is assigned a value corresponding to the amount of reflected light in the piece of continuous space, most often called pixel, which it represents. By, in addition, limiting the range of the image function to the interval of integers (the process is called quantization), we obtain a digital image, which we call a grey-level image.

Grey-level is a term generally used to describe the intensity of achromatic (monochromatic) light – light that is without colour. However, the human visual system can distinguish millions of different colour shades and intensities, but only around a couple of hundreds shades of grey. A great deal of extra information may be contained in the colour, and it is sometimes of interest to capture it in images. Colour depends primarily on the reflectance properties of an object, but also on the colour of the light source, and the nature of the human visual system. To describe any particular colour, not only the intensity (amount) of light is used, but also *hue* and *saturation*, which (together) determine the *chromaticity* of a given colour. Colour models have been created to specify description of a particular colour, by

Figure 1: The Risan Mosaic represents Hypnos, the God of Sleep. It was found in the second century A.D. building in the town of Risan in Kotor Bay, Montenegro. This image of the pagan God Hypnos is the only known image of this deity in the world.

identifying it as a point in a 3D coordinate system. As an example, in the colour model that is most often used for video cameras and monitors, called RGB model, a colour image consists of three independent image planes, one in each of the primary colours, red, green, and blue. A particular point in the colour image is assigned three values (coordinates) specifying the amount of each of the primary components present in it, in addition to two values indicating the spatial position of that point in the image plane.

Together with the development of science and technology, imaging has become a much more general term, related not only to capturing the reflectance of light, but also other physical properties of the objects of interest. The challenge became to make “images” of objects we cannot see, or “images” of new properties of objects (heat, density, water content, etc). Today, we can look at images of distant stars, atomic-size objects, parts of a living human body, unborn babies, blood flow. However, to be able to understand and interpret such images, the observer has to know what physical property is mapped to a grey-level image and how that property relates to the grey-levels present in the image. Consequently, image understanding and interpretation have become much more difficult. Some imaging techniques provide geometric information about the object (microscopy), some represent its anatomy and/or functionality (magnetic resonance angiography – MRA, positron emission tomography – PET), while others show topographic properties of an object (radars, ultrasound). The acquired data are organized in a way that preserves some spatial structure of the object of interest, even if that is not the main observed property of the object; the analogy is why the process of creating such data structures is called imaging, and data themselves – images. The spatial dimension of the image may in various ways correspond to the dimension of the imaged space. 2D images of 3D space may be projections (images of the visible surface of a 3D object), distance images, images of a 2D slice of a 3D object, or reconstructed images, obtained by tomography, when some information about internal density structure is derived from measurements in a number of line directions and incorporated into the 2D, or 3D, images of 3D objects. The most often used tomographic approaches are computed tomography – CT, PET, and MRI (magnetic resonance imaging). In addition, sequences of either 2D or 3D images over time are often of interest, where the additional – temporal – dimension is added to the existing spatial dimensions, and corresponding 3D and 4D images are created.

When extracting and analysing the information from the obtained images, we rely more and more on computers. They can handle huge amounts of data and accomplish some tasks, primarily those defined in terms of processing large sets of numerical parameters, much faster, and much more reliable than humans can. However, that requires developing computer vision systems, which is clearly a very difficult task. "Teaching computers to see" seems to be an ongoing challenge, but is still a constantly rewarding process. The application fields are numerous and the variety of tasks endless. Computerized image analysis is rapidly developing, supported by the development of technology and computer science and inspired by demands from a variety of application fields.

When developing image analysis methods, we are not focused on any particular method of grey-level image creation. No matter what physical property is imaged, it is practically never homogeneously exhibited. Its gradual change over the image, together with imprecision resulting from imaging conditions, like noise, or limited resolution, make it difficult to clearly "see" and separate different objects appearing in the image. The task of image segmentation is to extract the object(s) of interest, by selecting the points in the image which belong to the object. Image segmentation is considered to be both the most important and the most difficult task in image analysis. A decision if the point belongs to the object of interest, or not, is crucial for the quality of all following analysis steps and is often difficult to make. However, a segmentation which has a binary (two-valued) image as a result, where object points are mapped to "white", and non-object points (background) to "black", can hardly handle realistically uncertainties and heterogeneity of object properties. As a consequence, it becomes desirable to perform a fuzzy segmentation of an image. Such an approach allows points to belong to an object to some extent, and therefore avoids crisp decisions at this early analysis step. In this way, a larger amount of data is preserved, and can be used later in the process. The result of a fuzzy segmentation is a grey-level image of an object of interest, where object points are "white", background is "black", and grey-levels in between correspond to partial belonging of the points to the object, determined according to intensity, geometric, or other information available from the image.

In the development of methods for segmentation and further analysis of fuzzy images, we use knowledge from the theory of fuzzy sets. Necessities are briefly summarized in the next section.

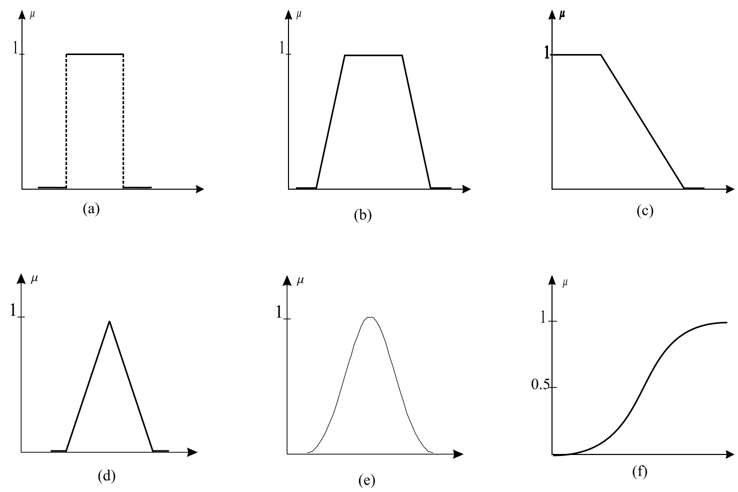


Figure 2: Families of parameterized functions which are commonly used as membership functions. Elements of a reference set are mapped to the set of real numbers and shown on the x-axis, while their memberships are given by μ. (a) Π-function. (b) Trapezoidal function. (c) Semi-trapezoidal function. (d) Triangular function. (e) Gaussian function. (f) S-function.

2.2 Fuzzy sets

A (continuous) fuzzy set is a collection of elements with a continuum of grades of membership; it is characterized by a membership function, which assigns a membership value between zero and one to each element. A fuzzy set is a generalization of a crisp set. While a crisp set either contains a given element, or it does not, which is described by the membership values one and zero, respectively, belongingness of an element to a fuzzy set can be partial, and is described by any value between zero and one. Some of the most commonly used membership functions are presented in Figure 2. The function in (a) is a Π-function, which is a membership function of a crisp set, and is also known as the characteristic function of a (crisp) set. The other presented functions can be used for fuzzification of any one-dimensional (1D) feature, where the assigned memberships reflect gradual changes of the feature values. When introduced by Zadeh (1965), the notion of a fuzzy set was expected to provide a starting point for the building of a conceptual framework, to exist in parallel with the framework of crisp ("ordinary") sets, but to be more general and potentially provide increased applicability in different fields, including image analysis. The framework was seen as a natural way of dealing with problems in which the source of imprecision is in the absence of sharply defined criteria for class membership, rather than in the presence of random variables. In other words, fuzzy set "large boats" can be created from, e.g., boats in a harbour as a reference set, not because the boats in the set are "probably large", but because they are "large to some (lower, or higher) extent".

Having in mind the difficulties in image segmentation, mainly caused by the existence of non-sharp boundaries between the objects in an image, it is not surprising that the comfortability of fuzzy sets, not forcing us to make hard decisions, became appreciated and well accepted in image analysis; for an overview of the applications, see Rosenfeld (1998). An object in the image is presented as a fuzzy set to which image points have graded memberships.

A fuzzy membership function is primarily defined as a mapping to the interval of real numbers $[0, 1]$ and is in general continuous. More formally, a *fuzzy subset* A of a reference set X is a set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where

$$\mu_A : X \rightarrow [0, 1]$$

is the *membership function of* A *in* X.

The (crisp) set of points having (strictly) positive memberships to the set A is called the *support* of A, while the *core* of a fuzzy set A contains the points with the membership to A equal to 1 (it is sometimes referred to as the *kernel*. When defined on a discrete domain, the membership function is a discrete function, and a corresponding set is a discrete fuzzy set. To represent an object in an image, we consider a fuzzy set defined on Z^2 or Z^3, being a typical space of discrete images. Such a set is called a *discrete spatial fuzzy set* (Bloch (2005)). When represented in a computer, the number of different membership values is finite, and limited by internal properties of the computer. Integer values are most often chosen to represent memberships, in order to increase the speed of computations. In this way, the range of a discrete fuzzy function is not the interval $[0, 1]$, but rather the set $\{0, 1, \ldots, \ell\}$. The value ℓ is often equal to 255, or 65 535, which corresponds to 8-, or 16-bit pixel depth (number of bits used to represent a pixel value).

The theory of continuous fuzzy sets is well developed. The development of discrete approaches requires adjustments of the existing results, but also creation of new ones. In our research, we have sometimes used the concept of *fuzzy step sets* (Bogomolny (1987)), which provides a link between continuous and discrete fuzzy membership functions. This type of sets is defined by a step-function, which takes $\ell + 1$ different values on its domain R^2, where R^2 is represented as a disjoint union of $\ell + 1$ open subsets, ℓ of which are bounded, with closures being rectifiable Jordan arcs and the only possible pairwise intersections of the observed (closed) subsets. The membership function is constant and positive on each of the bounded subsets, and is equal to zero on the unbounded one. We interpret a fuzzy digital image as a disjoint union of constant-valued image elements (pixels). Consequently, we can directly apply some of the results derived for continuous fuzzy sets to the discrete fuzzy sets we are primarily interested in.

However, a problem we are often facing, in connection with fuzzy image analysis, is that the membership function of an observed fuzzy object is not analytically defined. The membership values are derived from grey-levels of points, obtained in an imaging process, and sometimes, additionally, from a set of criteria designed to capture geometric, structural, and other properties of the imaged object. Results related to continuous fuzzy sets often rely on properties of an analytically defined continuous membership function, and cannot always easily be adjusted to the situation where the corresponding membership function is not analytically derivable. Consequently, we are most often forced to design methods appropriately applicable to discrete fuzzy sets and add new building blocks to the developing mathematical theories and algorithms for handling fuzzy discrete data appearing in digital images.

A representation of fuzzy sets which is often used as an alternative to representation by a membership function, is the one based on α-cuts. For a fuzzy set F, defined on a reference set X, the following two representations are equivalent (Dubois and Jaulent (1987)):

- a membership function $\mu_F : X \to [0, 1]$ which assigns to each $x \in X$ its membership grade $\mu_F(x)$ to the fuzzy set F;
- the set of α-cuts $C(F) = \{F_\alpha \mid \alpha \in (0, 1]\}$ of the set F, where $F_\alpha = \{x \in X \mid \mu_F(x) \geq \alpha\}$.

The connection between the membership function and the stack of α-cuts provides a common approach for fuzzification of crisp functions. The fuzzification principle, based on one of the following equations:

$$f(S) = \int_0^1 \hat{f}(S_\alpha) d\alpha, \tag{1}$$

$$f(S) = \sup_{\alpha \in (0,1]} [\alpha \hat{f}(S_\alpha)] \tag{2}$$

can be used to fuzzify a binary function $\hat{f}$. In this way, various properties defined for binary sets (i.e., α-cuts) can be generalized to fuzzy sets, including the membership function itself; if the characteristic functions of the α-cuts are observed, the membership function of the corresponding fuzzy set can be obtained by either their integration, or by taking the supremum of their weighted values, over the "height" of the stack.

The other approach to derive fuzzy definitions from crisp ones is to translate binary set theoretical and logical operations and relations into fuzzy equivalents. The fuzzification is performed by, first, replacing the notion of a set by a notion of a membership function, and then by fuzzifying negation (set complement), conjunction (set intersection), and implication (set inclusion), as the basic logic/set operations. Operators called negator, conjunctor, and implicator, respectively, can be used for fuzzification (Dubois and Prade (1980)):

Negator is a unary operator on the interval $[0, 1]$, which coincides with the Boolean negation on $\{0, 1\}$. It can additionally be required to be a decreasing and involutive mapping. An often used negator is defined by $\mathcal{N}(x) = 1 - x$.

Conjunctor is a binary operator on the interval $[0, 1]$, which coincides with the Boolean conjunction on $\{0, 1\}^2$. An operator most often used as a conjunctor

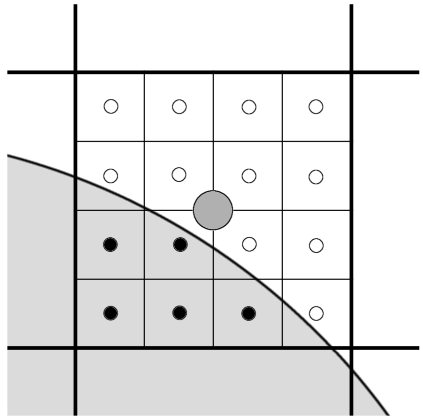


Figure 3: The discrete approximation of area coverage fuzzification facilitated by super-sampling of (border) pixels. The membership of the presented pixel (divided into 16 sub-pixels) to an object is equal to $\frac{5}{4^2}$.

is a t-norm. The properties of a t-norm are (1) it is an increasing mapping in each variable, (2) for each $x \in [0,1]$ it holds that $\mathcal{C}(1,x) = \mathcal{C}(x,1) = x$, (3) it is commutative, (4) it is associative. Often used conjunctors are defined by $\mathcal{C}(x,y) = \min\{x,y\}$, or $\mathcal{C}(x,y) = x \cdot y$. By duality, a t-conorm is derived from a t-norm. It is often used to define disjunctors, binary operators on the interval $[0,1]$, which coincide with the Boolean disjunction on $\{0,1\}^2$. A t-conorm is increasing in each variable, commutative, associative, and has 0 as the neutral element.

Implicator is a binary operator on the interval $[0,1]$, which coincides with the Boolean implication on $\{0,1\}^2$, and is a decreasing mapping in the first, and an increasing mapping in the second variable. In addition, for each $x \in [0,1]$ it holds that $\mathcal{I}(1,x) = x$. Often used implicators are defined by $\mathcal{I}(x,y) = \min\{1, 1-x+y\}$, or $\mathcal{I}(x,y) = \max\{1-x, y\}$.

Other set theoretic operations are easily derived from the operations listed above.

It is important to mention one additional fuzzification approach, the one most often appearing in image analysis: fuzzification resulting from an imaging process, and determined by the properties of the imaging device. It is often practically impossible to analytically express a specific combination of characteristics of an imaging device and conditions resulting from additional knowledge used in assigning memberships to the points. However, it is very important to know which property is imaged and what kind of correspondence between the observed property and resulting grey-levels in the image is exhibited, and to incorporate that knowledge in any specific image analysis technique.

Many imaging devices produce grey-level images with grey-levels corresponding to the area of the pixel in the grid, covered by the imaged object. This is called *area coverage fuzzification*. In this case, grey-levels conveniently reflect the grade of belongingness of an image element to the object and a fuzzy discrete representation of the object is immediately provided (by rescaling, memberships can be mapped to the interval $[0, 1]$). Area coverage fuzzification is often used in our studies. In our theoretical work, area coverage of a pixel is expressed as the number of sub-pixels within the candidate pixel which have their centroids inside the (continuous) object. This is facilitated by super-sampling of pixels located around the object border, as illustrated in Figure 3. In this example, the centres of five (black) out of 16 sub-pixels, obtained by super-sampling of an observed pixel by factor four, are covered by an object. These five sub-pixels contribute to the fuzzy membership of the observed pixel to the object, which is then equal to $\frac{5}{4^2}$. The same pixel does not belong to the crisp digitization of the observed object, since its centre, marked by a grey dot, is not covered by the object.

3 Segmentation and defuzzification

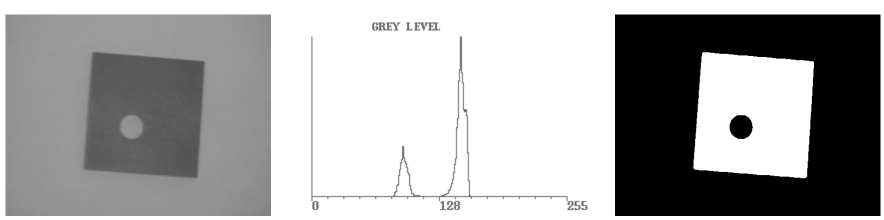
Image segmentation is a process of partitioning an image into (several) areas containing uniform feature characteristics. It is one of the first and most difficult tasks of any image analysis process. Whatever the subsequent steps may be, object detection, feature extraction, object recognition, scene interpretation, or image coding, they rely heavily on the quality of the segmentation process.

There are many different segmentation techniques proposed in the literature, see e.g., Gonzalez and Woods (2002) and Sonka et al. (1998). All of them are based on (one of) two basic properties of the grey-levels in an image: similarity and discontinuity. According to that, segmentation algorithms can be roughly classified into two categories; they can be *region-based* or *boundary-based*.

Region-based segmentation relies on some similarity property. Its goal is to split the image into disjoint sets of connected pixels, similar according to a homogeneity criterion. Boundary-based algorithms use the discontinuity property to detect the boundaries of objects within the image. In many cases, the segmentation performance can be improved by utilizing combinations of the two approaches.

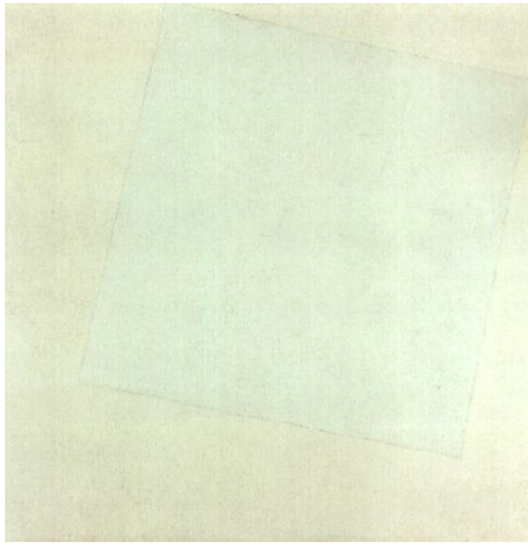
3.1 Crisp segmentation

The goal of crisp segmentation is to subdivide an image into crisp regions. The simplest and most popular crisp segmentation technique is *grey-level thresholding*, a process of partitioning image pixels into two regions — object and background — by identifying a threshold value such that the grey-levels of the object points are greater than or equal to the threshold, and the grey-levels of the background points are less than the threshold, or vice versa. Thresholding is a special case of pixel-wise classification where the decision whether a pixel belongs to an object or not is based solely on the grey-value of the pixel. There exists a large number of methods for selecting suitable threshold values; for an overview, see e.g., Gonzalez and Woods (2002). The most popular ones are based on an analysis of the grey-level histogram of the image. The histogram (Figure 4(b)) of the grey-level image presented in Figure 4(a) shows a bi-modal distribution. The image can be segmented by thresholding at the grey-level 120; the result is shown in Figure 4(c).

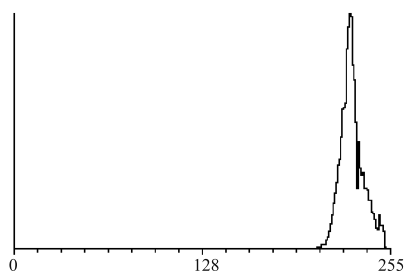


(a) (b) (c)

Figure 4: (a) Grey-level image. (b) Grey-level histogram of (a). (c) Crisp segmentation of (a) based on thresholding.



(a)



(b)

Figure 5: (a) Kasimir Malevich (1878–1935), *White square on white background*. (b) Intensity histogram of (a).

A drawback of this type of methods is that they take into account neither information from neighbourhood of a point, nor shape properties of the objects. Segmentation by thresholding would not give good results if applied to, e.g., image in Figure 5(a). The histogram of a grey-level converted version of the image is shown in Figure 5(b). It is clear that not many segmentation methods would perform well on this image. The shape information is probably essential in this case.

Region-based techniques assume partitioning of an image into a disjoint union of connected regions, such that some property is satisfied for the points in each of the regions, but not in any union of the regions. Partitioning can be performed by region growing, or by region splitting and merging.

Region growing is a procedure which groups points into regions, starting from a set of seed-points and iteratively adding to that set its neighbouring points with similar properties. Criteria related to size and shape of the object, or similarity of the pixel to add and the region grown so far are often used to design the growing process.

The region splitting and merging approach starts from an arbitrary subdivision of an image into disjoint regions, instead from seed-points. By iterative divisions and groupings, regions satisfying predefined criteria are obtained, while an optimal number and size of regions are (preferably) achieved.

Boundary-based techniques are based on detection of grey-level discontinuities in images, indicating (isolated) points, lines, and edges. An edge is a set of connected pixels that lie on the boundary between two regions, which is, ideally, indicated by a vertical step transition in grey-levels between two regions. However, in practice, the transition is most often blurred; consequently, it may be hard to detect relevant edges. Edge-linking is the next step performed in order to detect objects' boundaries; both local and global techniques can be used for that.

Segmentation based on watersheds is a widely used method that combines concepts from both region- and boundary-based approaches, see e.g., Vincent and Soille (1991). In a region growing style, starting from local minima as seeds, regions are simultaneously created and the boundaries between them are set at positions that prevent region merging. Detected boundaries therefore reflect certain discontinuities in the image.

Crisp segmentation methods are numerous; let us mention, in addition, deformable shape models (snakes), template matching, and level sets, as very popular ones. Unfortunately, all of them lead to a significant loss of information contained in a grey-level image, when a crisp dichotomization of pixels is performed. Crisp segmentation techniques generally do not preserve the inherent uncertainties associated with real images and also fail to overcome noise, blurring, and background variation.

It seems that many of these problems could be solved by introducing fuzzy, instead of crisp, segmentation.

3.2 Fuzzy segmentation

The process of converting the input image into a fuzzy set by indicating, for each pixel, the degree of its membership to the object, is referred to as *fuzzy segmentation*. The most straightforward way to perform fuzzy segmentation is to scale grey-levels of an image to be between zero and one; in that way the grey-level of a pixel can be seen as its membership to the set of high-valued (bright) pixels. If the image contains a bright object on a dark background, such a scaling produces the membership function of the object (the degree of belongingness of a pixel to the object). In most cases, though, more advanced segmentation methods are required, especially since it is rarely sufficient to use only the brightness of pixels to calculate fuzzy membership values.

The work presented in this thesis, being related to the development of shape analysis methods, is intended to be independent of any particular fuzzy segmentation method, i.e., applicable to any fuzzy image. Therefore, segmentation is not specially analysed, or discussed, in the included papers. However, we would like to emphasize that the theory of fuzzy sets and fuzzy logic show their great power particularly in image segmentation. Not only the nature of fuzzy sets, allowing partial memberships, provides a natural way of representing objects with no sharply defined borders, but also fuzzification of features relevant for object definition, and their combination by using rules of fuzzy logic, provide reliable final conclusions even in the case of contradicting or vague input features. Consequently, it becomes much easier to incorporate and combine different pieces of information regarding both imaged objects and imaging devices, and also to overcome the presence of noise, or other undesired imaging effects, much more successfully than by conventional (crisp) segmentation techniques.

In the following, two fuzzy segmentation methods are briefly described. These are two particular methods applied to create objects that we have used in our studies on shape analysis methods. In addition, we consider them to be good representatives that illustrate the main properties and the power of fuzzy segmentation methods. They are both region-based and are known as *fuzzy thresholding* and *fuzzy segmentation based on connectedness*.

3.2.1 Fuzzy thresholding

Thresholding is a segmentation method appropriate to use when the object and the background are separable by grey-levels alone. In that case, the grey-level histogram is often characterized by two modes and a suitable threshold is a value corresponding to a valley point between the two modes.

It is often difficult to identify a unique threshold for crisp dichotomization of pixels. Therefore, several "fuzzified" thresholding schemes are proposed; Jawahar et al. (1997, 2000) give an overview.

Fuzzy thresholding is a process of partitioning an image into two fuzzy sets, corresponding to the object and the background regions, by identifying the membership distributions associated with them. There are different ways to define the membership functions; some of them reflect the ambiguity in the transition region between the object and the background classes, whereas others depict the geometric structure of the object and the background grey-level densities. In both cases, the regions are no longer guaranteed to be mutually exclusive.

In Figure 6, membership distributions obtained by different algorithms are shown. In all the cases, the memberships are assigned so that the fuzziness in the image is minimized. The average amount of fuzziness in an image is expressed by the index of fuzziness, which measures a distance (which can be defined in several ways) in each point between its fuzzy membership value and the closest crisp membership, and takes the average. Other optimizations of fuzziness, like, e.g., minimizing entropy, or maximizing the index of crispness, are also possible to use for the same purpose (Pal and Rosenfeld (1988)). However, it should be noted that using any of the mentioned criteria (index of fuzziness, index of crispness, and entropy) assumes that the best segmentation result is a crisp one. This assumption excludes intrinsic fuzziness possibly present in the image.

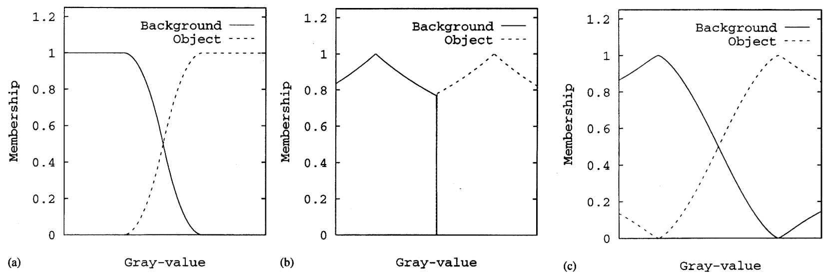


Figure 6: Membership distributions assigned using (a) Murthy and Pal (1990) (b) Huang and Wang (1995) (c) Fuzzy c-means (Bezdek (1981)) algorithms.

The algorithm suggested in Murthy and Pal (1990) identifies the fuzziness in the transition region between the object and the background classes. The membership value of a point to the object is determined by applying the S-function to the grey-levels in an image:

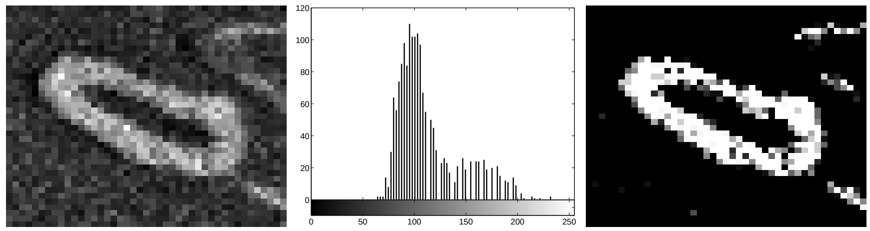
$$S(x;a,b,c) = \begin{cases} 0, & x \leq a; \\ 2\left(\frac{x-a}{c-a}\right)^2, & a < x \leq b; \\ 1 - 2\left(\frac{x-c}{c-a}\right)^2, & b < x \leq c; \\ 1, & x > c, \end{cases} \tag{3}$$

where $b = \frac{a+c}{2}$. The value b is the cross-over point of the fuzzy set defined by $S(x;a,b,c)$, i.e., the membership value of $x = b$ is equal to 0.5. For a range of cross-over points and the fixed band-width $\Delta b = b - a = c - b$, the S-function is applied to the original grey-levels, and the index of fuzziness is calculated. The cross-over point providing the lowest index of fuzziness is taken as a threshold. The choice of Δb is made in accordance with the original grey-level distribution. The background region is determined as the complement of the object region.

The approach taken in Huang and Wang (1995) and the approach using fuzzy c-means clustering (Bezdek (1981)), determine the membership value of a point considering its grey-level, with respect to the mean grey-level value of a region (object or background).

The object and background regions obtained by the algorithm suggested in Huang and Wang (1995) are fuzzy sets with mutually exclusive supports; a point is assigned either to the object, or to the background, with the membership between 0.5 and 1 (the closer the intensity of a point to the mean of the region, the higher its membership to that region). The threshold is chosen in a way that minimizes the fuzziness in the image, expressed by, e.g., entropy.

The fuzzy c-means classification algorithm (Bezdek (1981)) can also be used to perform fuzzy thresholding if the fuzzy c-means objective function is adjusted



(a) (b) (c)

Figure 7: (a) Grey-level image; (b) its grey-level histogram; (c) fuzzy segmentation obtained by two-level thresholding, at levels 130 and 160, while keeping a fuzzy border of the object. The object is a wood fibre, imaged in an electron microscope.

so that the number of clusters is $c = 2$ (object and background), the number of elements to be classified is equal to the number of different grey-levels, and the membership of an element to a cluster is expressed in terms of the frequency of occurrence of a grey-level j and the membership value of j to each cluster. For each cluster, the cluster mean (prototype) is taken to be the mean grey-level of the region. Such an objective function is iteratively minimized by updating the cluster means and memberships. The memberships of each point to the object and to the background sum up to one, i.e., object and background regions complement each other.

The thresholding method moth often used in our studies is the one suggested in Murthy and Pal (1990). We have sometimes simplified it by utilizing the fact that for objects in our focus the points are usually not difficult to classify into three classes, by setting two thresholds. The points which are most certainly inside the object (above the higher threshold) belong to one class and are assigned membership one. Points which are most certainly background points (below the lower threshold) are in the second class and are assigned membership zero. The transition between these two classes, with grey-levels between the two thresholds, is kept as a *fuzzy border* of the object, with memberships of the points obtained by linear rescaling the grey-levels to values between zero and one. This fuzzy segmentation method, appealing due to its simplicity, is appropriate in situations where grey-level distributions of an object and a background are partly overlapping, which can, e.g., be assumed if the grey-level histogram of the image is bi-modal. An example is shown in Figure 7; a cross section of a wood fibre in a scanning electron microscope is presented in (a), and the image histogram is given in (b). The modes of the histogram (after being convolved with a suitable kernel) are at 100 and at 175, approximately. The distributions, assumed for the object and the background in accordance to the shape of the histogram, overlap between grey-levels 130 and 160, approximately. A fuzzy segmentation of a fibre is presented in (c), where the pixels with grey-levels lower than 131 are assigned 0 (background), those with grey-levels higher than 159 are assigned 1 (object), while the pixels with grey-levels between 131 and 159 are linearly rescaled to values between 0 and 1 and kept as a fuzzy border of the object.

3.2.2 Segmentation based on fuzzy connectedness

Image segmentation can be performed by a region-growing procedure which determines an object as a maximal connected component of an image satisfying certain properties. This approach relies on the basic concepts of digital topology, such as adjacency and connectedness. In order to extend this segmentation method to the fuzzy case, it is required to first extend some of the topological concepts needed. Early results are presented in Rosenfeld (1979), where a fuzzy segmentation procedure based on topological features of fuzzy subsets is introduced. This work is further developed in Udupa and Samarasekera (1996) and Saha et al. (2000).

As in the crisp case, *a path* between two points in a (crisp or fuzzy) digital image is defined as a sequence of points such that every two successive points are adjacent and the end-points of the sequence are the two pre-determined points.

A general definition of *adjacency*, as any reflexive and symmetric fuzzy relation, independent of any image information, but only characterizing the underlying digital grid, is given in Udupa and Samarasekera (1996). (Let us note that an n-ary fuzzy relation on X is a fuzzy subset on X^n.) It is expected to be a non-increasing function of the distance between points, i.e., the closer the points are, the more adjacent they are to each other. This definition includes adjacencies commonly used in crisp digital images. Even though they are all crisp relations, strictly distinguishing if two points are adjacent or not, all of them can be used for defining adjacency between points in a fuzzy set. One example of fuzzy adjacency function n_{PQ} between two points P and Q in the image space, given in Bloch (2005), is

$$n_{PQ} = \frac{1}{1 + d(P, Q)},$$

where $d(P, Q)$ denotes the Euclidean distance between P and Q in S.

The strength of the path is defined as the strength of its “weakest link”. While Rosenfeld (1979) considers the “weakest link” to be the point with the smallest intensity (membership value) of all the points in the path, Udupa and Samarasekera (1996) and Saha et al. (2000) express the strength of a link in terms of fuzzy affinity. A *fuzzy affinity* is a fuzzy relation which combines adjacency, as a purely spatial concept, with two components based on the intensities of the points — a homogeneity based component and an object-feature based component.

The *homogeneity* based component reflects the degree of local “hanging-togetherness” (intuitively recognizable belongingness to the same object) of the image elements on account of similarity of their intensities. A function which can be used to express the degree of homogeneity of a pair of image elements is, e.g., the semi-trapezoidal function (Figure 2(c)) applied to the absolute value of the difference between the grey-levels of the observed points. It assigns membership one to the pairs of points with grey-levels which differ less than some threshold and membership zero to the pairs of points which are “less homogeneous” than some other threshold. In-between homogeneities of pairs are described by monotonously decreasing values between one and zero.

The *object-feature* based component reflects the degree of "hanging-togetherness" of points due to the similarity of their intensity values to some (specified) intensity, taken to represent the particular object which should be segmented. A function which can be used to assign the degree of fulfilment of this criterion to an image element is, e.g., trapezoidal membership function (Figure 2(b)) with the modal value equal to the intensity of the specified feature. The function is applied to the grey-levels of the points and assigns membership value 1 to those which are "close enough" to the observed value, whereas the membership decreases for the points whose grey-levels differ more.

In order to define fuzzy connectedness as a global (fuzzy) relation, every possible pair of grid points is observed. The strength of connectedness is assigned to each pair of points and determined as the strength of the strongest path between them.

The relation "to be connected" defined in Rosenfeld (1979) is the crisp relation; two points are connected if the strength of connectedness between them is not lower than the minimum of their intensities. Such an approach provides a possibility to keep the definition of an object to be the same as the classical (crisp) approach: an object is the maximal connected component of an image. In order to extract maximal (fuzzy-)connected components, it is suggested to start from the connected regions of points with the highest (constant) grey-level and to detect all of the image points fuzzy-connected to such a region. Each (maximal) region extracted this way is a fuzzy object. Let us observe that the image points can have positive membership to more than one object, i.e., objects may overlap.

If connectedness is defined to be a fuzzy relation, instead of defining an object as a maximal connected component of an image, it is possible to introduce some "tolerance", and define a fuzzy object as a fuzzy connected component of an image of a given strength, as it is done in Udupa and Samarasekera (1996) and Saha et al. (2000). A fuzzy object is defined as a fuzzy subset of an image such that the strength of connectedness of any two of its points is higher than the given strength.

We use the approach suggested in Rosenfeld (1979) to determine the strength of connectedness of each point in an image to the core of the fuzzy set. By thresholding at a relatively low strength of connectedness, we perform fuzzy segmentation in the image examples shown in Paper VII.

3.3 Defuzzification

In spite of many advantages of fuzzy segmented images, a crisp representation of objects may still be needed in the end. Reasons for that are, e.g., to facilitate easier visualization and interpretation. To visually interpret fuzzy structures in an objective way is a difficult task. Even though it contains less information, a crisp representation is usually easier to interpret and understand, especially if the spatial dimensionality of the image is higher than two. Moreover, analogues for many tools available for the analysis of binary images are still not developed for fuzzy images, which may force us to perform at least some steps in the analysis process by using a crisp representation of the image. In Paper VII, and to some extent in Paper III, we were interested in generating a crisp representation of a continuous (crisp) original object by utilizing the information that can be derived about it from its fuzzy discrete representation. The process of replacing a fuzzy set by some appropriately chosen crisp representative is referred to as *defuzzification*. We used defuzzification of a fuzzy segmented object as an alternative to a crisp segmentation of a grey-level image. The main advantage of this method is that it provides the possibility to incorporate the knowledge about the object to be segmented, which is not necessarily available in advance, but can be derived from a fuzzy representation, and can be used in the crisp segmentation process.

In literature, defuzzification is most often seen as the reduction of a fuzzy set to a (crisp) point; this approach is mostly used in fuzzy control systems. The best known method, among several existing in the literature and practise, to reduce a fuzzy set to a point is the *Centre of gravity*, which reduces the fuzzy set to its centre of gravity, defined as

$$\text{CoG}(S) = \left(\frac{\displaystyle\int_X x \cdot \mu_S(x,y)\ dxdy}{\displaystyle\int_X \mu_S(x,y)\ dxdy}, \frac{\displaystyle\int_X y \cdot \mu_S(x,y)\ dxdy}{\displaystyle\int_X \mu_S(x,y)\ dxdy} \right), \tag{4}$$

for a fuzzy set S of $X \subset R^2$. An analysis and evaluation of several other widely used defuzzification techniques is presented in, e.g., Leekwijck and Kerre (1999).

The defuzzification process is often decomposed into two steps: first, replacement of a fuzzy set with a crisp set, and second, reduction of a crisp set to a single point, as suggested in Ogura et al. (2001). The first step in the defuzzification is studied in, e.g., Roventa and Spircu (2003), where a selection of criteria, which should be fulfilled in the process of replacing a fuzzy set with its crisp counterpart, is formulated. We are primarily interested in defuzzification to a set; applied in image processing, it corresponds to replacing a fuzzy image by a binary image.

In image processing, the selection of a specific α-cut is, in most cases, used as the defuzzification method. This is essentially not more than thresholding a fuzzy image. How to choose the most appropriate value of α to perform defuzzification by thresholding is studied in, e.g., Udupa and Samarasekera (1996); Jawahar et al. (2000); Pal and Rosenfeld (1988); Pal and Ghosh (1990). An alternative approach for defuzzification is to incorporate fuzzy set theory in the merging criteria for various region growing segmentation methods. In that case, defuzzification is performed for each point to be merged, in order to make a (crisp) decision whether the point should be included in the region or not (Saha and Udupa (2001); Steudel and Glesner (1999)).

Our opinion is that the most appropriate approach to perform defuzzification in image processing is to combine two views, (1) where defuzzification is seen as an inverse of fuzzification and aims to recover the fuzzified original by deriving and applying the inverse of a fuzzification function to the fuzzy set, and (2) where defuzzification is a mapping from the set of fuzzy sets to the set of crisp sets, defined independently of any crisp fuzzified original that should be recovered. The two approaches are illustrated in Figure 8. By combining them, we try to overcome the

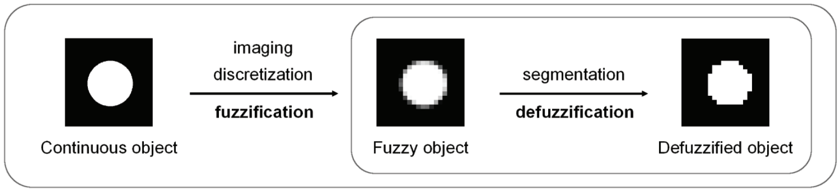


Figure 8: Two views of defuzzification: as a mapping from the set of fuzzy sets to the set of crisp sets, and as an inverse to fuzzification.

lack of information about the fuzzification function, as it is most often unknown in image processing. In Paper VII, we design a defuzzification function based on the properties of fuzzy sets, at the same time providing recovering of the original object.

4 Shape analysis

The shape of an object is a representation of its extent. It can be thought of as a silhouette of the object. It is often referred to as a region.

The shape of an object is invariant to geometric transformations such as translation, rotation, (uniform) scaling, and reflection. Therefore, shape can be understood as an equivalence class in the set of objects; two objects are equivalent (i.e., have the same shape) if there exists a series of translations, rotations, scalings, and reflections that maps one of them to the other.

There are many situations where image analysis can be reduced to the analysis of shapes. Shape analysis is an integral part of most applications of image analysis, being the major task in some of them, e.g., in robot navigation or optical character recognition (OCR). In our work, presented in Papers I–VI, well-known methods for analysis of crisp shapes, based on (1) moments, (2) signature, and (3) geometric descriptors of a shape, are generalized and applied to the analysis of fuzzy shapes. In this section, we give a brief description of these crisp techniques.

4.1 General aspects of shape analysis

There exist different classifications of shape analysis techniques, see, e.g., Loncaric (1998) for an overview. Depending whether only the shape boundary points are used for the description, or alternatively, the whole interior of a shape is used, the two resulting classes of algorithms are known as boundary-based (external) and region-based (internal), respectively. Another criterion for the classification of shape analysis techniques is based on different mathematical means used to describe a shape (Kidratenko (2003)); according to it, the functional approach corresponds to the boundary based one, whereas the set theoretical approach corresponds to the region based one. Examples of the former class are algorithms which parse the shape boundary and various Fourier transforms of the boundary. Their main advantages are reduction of data and (in some cases) a convenient description of complex forms. Region-based methods include, e.g., the medial axis transform, moment-based approaches, and methods of shape decomposition into the primitive parts. Their main advantages are easier characterisation and stability in practical applications, where there is unavoidable noise.

A *description* of a shape is data representing it in a way suitable for further computer processing. Such data can be low-dimensional (perimeter and moments), or high-dimensional (medial axis and primitive parts). The first type of data is suitable for, e.g., shape classification, while the second, often called shape *representation*, provides good visual interpretation and compression.

A classification scheme for shape analysis methods may look as follows (Loncaric (1998)):

Boundary based numeric methods result in a numerical description based on shape boundary points. An example of this approach is perimeter (understood as the length of the boundary of a 2D shape, and not as the boundary itself), but also one-dimensional functions constructed from the two-dimensional shape boundary, called shape signatures. In that case, the shape is described indirectly by means of a one-dimensional characteristic function of the boundary, instead of the two-dimensional boundary itself. The Fourier transform is often applied to the signature functions, and used as a shape descriptor.

Boundary-based non-numeric methods take shape boundary as input and produce the result in a pictorial or a graph form. Examples are boundary approximations by polygons and splines, and boundary decomposition.

Region-based numeric methods compute scalar result(s) based on the global shape. Moment-based methods are popular examples from this group. Area and compactness measures are also often used, although not information-preserving; it is not possible to recover the shape if only its area, or its compactness measure, are known.

Region-based non-numeric methods result in a spatial representation of a shape, based on the whole shape's interior. The most popular methods in this group are the medial axis transform and the shape decomposition. Mathematical morphology, suitable for shape-related processing, since morphological operations are directly related to object shape, belongs to this group of approaches as well.

The goal of a shape description is to uniquely characterize the shape. The desired properties of a shape description scheme are invariance to translation, scale, and rotation; these three transformations, by definition, do not change the shape of an object, and consequently should not change its descriptor. However, it should be noted that in the discrete case such invariance exists only up to discretization effects, and often special care must be taken in order to fulfil it.

Additional desired properties of a good shape description method are (Loncaric (1998)):

- accessibility – How easy is it to compute a descriptor in terms of memory requirements and computational time; are the operations local or global?
- scope – How wide is the class of shapes that can be described by the method?
- uniqueness – Is the representation uniquely determined for a given shape?
- information preservation – Is it possible to recover the shape from its descriptor?
- stability and sensitivity – How sensitive is a shape descriptor to small changes of a shape?

Our research, presented in this thesis, has been related to three different shape description methods. We studied moments of a shape (Papers I–III), the signature of a shape based on the distance from the shape centroid (Paper IV), and 2D and 3D geometric shape descriptors – area, perimeter and compactness measure of a shape, as well as volume, surface area, and sphericity measure (Papers V and VI). The main intention has been to extend these well-known descriptors and to apply them to the analysis of discrete spatial fuzzy sets. In the following, we briefly present our starting points regarding each of the shape analysis methods observed, before we give an overview of our contributions to the field in Section 5.

4.2 Moments of a discrete shape

The two-dimensional Cartesian moment, $m_{p,q}$ of a function $f(x, y)$ is defined as

$$m_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)x^p y^q \; dxdy \; ,$$

for integers $p, q \geq 0$. The moment $m_{p,q}(S)$ has the order $p+q$. Cartesian moments are often referred to as *geometric moments*.

A complete moment set of order n consists of all moments $m_{p,q}$ such that $p+q \leq n$. The geometric moment $m_{p,q}$ can be seen as the projection of $f(x, y)$ to the monomial basis set $x^p y^q$.

Moments were introduced into image analysis by Hu (1962). Hu stated that for a piece-wise continuous function $f = f(x, y)$, non-negative only within a bounded set in R^2, the moments of all orders exist. Moreover, the set of moments of a function f is uniquely determined by f, and conversely, the set of all moments of f uniquely determines f. Hu's statement holds for image functions, since they are bounded and can be interpreted as piece-wise constant (i.e., step-) functions.

In order to apply geometric moments as shape descriptors, their behaviour under scaling, translation, rotation, and reflection are studied. To provide invariance of a shape descriptor to scale, translation, and rotation of a shape, Hu defined seven nonlinear combinations of geometric moments up to order three, which are known as the *absolute moment invariants*.

When used in image analysis, moments are calculated for discrete functions on discrete bounded domains. The definition of a geometric moment $m_{p,q}$ of a digital image $f(x, y)$ is

$$m_{p,q} = \sum_i \sum_j f(i, j) i^p j^q \; ,$$

where (i, j) are points in the (integer) sampling grid.

An image is always bounded and an image function is piece-wise continuous, which ensures that the Hu's statement holds for digital images. However, potentially huge number of moments (as many as there are pixels in the image itself) may be required for a unique representation and reconstruction of a digital image. To make the description practically feasible, a finite set of moments has to be used, and consequently, only an approximate reconstruction can be provided. The important question is, therefore, how to make an appropriate selection of moments, such that sufficient information is provided for a unique characterization of the image.

It was noticed already by Hu that, in some experiments, object descriptions by moments were very similar for rather different shapes, and that the size of variation of shapes was not consistent with the size of variation in the description. This was, quite naturally, seen as a consequence of a limited resolution of the image, as well as the limited number of moments involved in the description. It is clear that the image resolution has a large influence on the quality of the description by moments. That is confirmed by, among others (Klette and Žunić (2000); Teh and Chin (1986)), our results presented in Papers I–III. It is, however, not easy to fully understand and predict the influence of the behaviour of high-order moments in the image description. If the image function is described by integer values in the integer grid, the corresponding moments are integer-valued, which is advantageous, since floating-point calculations and, consequently, rounding errors, are avoided. Unfortunately, in that case, higher order moments increase in size very rapidly, and are rather sensitive to noise, which gives reasons to try to avoid using them in the description. If, alternatively, the rescaling of image is performed, and the domain (and consequently, the basis monomials) is restricted to the interval $[-1, 1]$, the monomials are highly correlated and the important information is contained within a small difference between them. This problem is studied in Talenti (1987), where it is concluded that (even in the continuous case) the problem of reconstruction of a function from its (infinite) set of geometric moments (known as the Hausdorff moment problem) is ill-posed, i.e., the required inverse is not continuous; an arbitrarily small change of "input" moment values can produce unpredictable change of the reconstructed function. This has been found to be a consequence of the non-orthogonality of the basis monomials $x^p y^q$ and has initiated several alternative approaches. Polynomials of an orthogonal basis set are used for moment definitions (e.g., Legendre, Zernike), in order to provide a stable and simple reconstruction. For an overview, see Prokop and Reeves (1992).

Still, geometric moments have become well accepted shape descriptors, due to their simple definition, their uniqueness for a given shape, the possibility to derive descriptors invariant to rotation, translation, and scaling, and to express them as integers, their linearity, and the possibility to reconstruct a number of features of a shape from an appropriately chosen set of its moments. In addition, it is possible to express all other types of moments in terms of geometric moments.

The disadvantage of sensitivity to noise mostly applies to high-order moments. Their use in object description can be avoided (or, at least, reduced) by, e.g., first decomposing complex shapes to simpler and more regular parts, which can, then, be represented by a smaller set of the corresponding moments. This approach was a motivation for our studies performed for disks, which are among the most commonly appearing shapes in images. We have analysed representation and reconstruction of both crisp and fuzzy disks by the set of their appropriately chosen moments.

Our focus was on the analysis of the errors of estimation of moments of a continuous shape from the corresponding moments of its (crisp or fuzzy) digitization. The interest for such research was emphasized in, e.g., Teh and Chin (1986); Liao and Pawlak (1996). Our theoretical results, primarily derived in Paper III, confirm empirical studies presented in Teh and Chin (1986). Paper I is devoted to the properties of representation and reconstruction of crisp (digital) ellipsoids (spheres, ellipses, and disks are observed as special cases) from the set of shape moments up to order two. Papers II and III are focused on fuzzy spatial sets. In Paper II we study the objects fuzzified by the area coverage approach, and relate fuzzy and spatial resolution. In Paper III, we observe a fuzzy digital disk with a membership function defined by a non-increasing function of the distance of a point to the chosen central point. Various aspects of the shape representation by moments are considered in both cases, and advantages of using fuzzy spatial sets are confirmed.

4.3 Shape signature based on the distance from the centroid

A *signature* is a one-dimensional functional representation of a two-dimensional shape boundary. The simplest way to generate a signature is to traverse the boundary and plot the distance from the centroid (or, alternatively, some other point of special interest) to the boundary as a function of the central angle, starting from some reference line, see Figure 9. This function is also called the radius-vector function (Kidratenko (2003)).

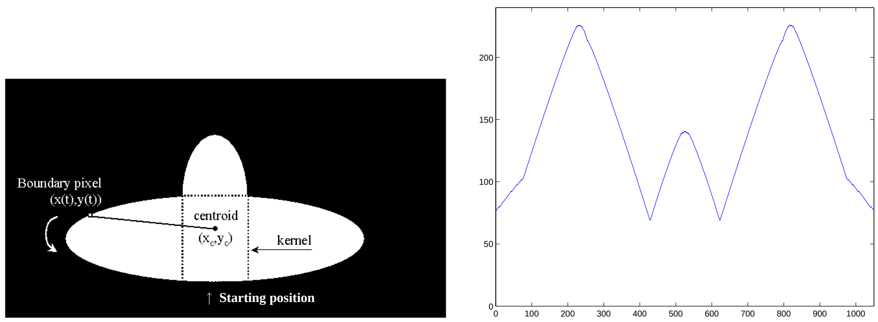


Figure 9: A star-shaped object with respect to the centroid and its corresponding shape signature.

In order to provide uniqueness and recoverability of this shape descriptor, it is necessary that the object is *star-shaped* with respect to the chosen central point. The object is star-shaped with respect to a point O, if for any point P on the object boundary, the line segment connecting P and O is entirely contained in the object. The set of all points with respect to which the object is star-shaped is called the *kernel* of the star-shaped object. If the central point of a shape belongs to the kernel, the radius-vector function uniquely describes the shape. Hence, the unique reconstruction is possible.

The radius-vector function of a continuous star-shaped object is invariant under translation, but depends on scaling, rotation, and reflection (Kidratenko (2003)).

In the discrete case, however, even translation affects the signature of a shape. In addition, a selection of boundary points to be used for the representation by the radius-vector function has to be done. The points can be chosen in several ways and the selection, in general, affects the description. Some possible ways to choose the boundary points for representation are, e.g., (1) to take equidistant points on the boundary, (2) to use vertices of a polygonal approximation of the boundary, (3) to select points where the curvature of a boundary is high. The most common way is to take the boundary points so that the central angles are equal.

Our study on shape signature (Paper IV), based on the distances from the shape centroid, was performed with the intention to reduce the sensitivity of this shape descriptor to translation of the shape centroid within a pixel, by using fuzzy, instead of crisp, representation of shapes.

4.4 Geometric descriptors

In many applications of digital image analysis, quantitative geometric measures of the imaged continuous objects are of great interest. The information about perimeter and area of 2D objects, as well as surface area and volume of 3D objects, is often required. If obtained from digital images, these measurements are only estimates of the true object properties. The quality of the estimates is expressed in terms of accuracy and precision, but also in terms of computational demands, algorithmic complexity, and sensitivity to noise. Various approaches are known in the literature, related to both crisp and grey-level objects. Area, perimeter, and compactness measure (in 2D), and volume, surface area, and sphericity (in 3D) are considered in the following and referred to as *geometric descriptors*.

4.4.1 Geometric descriptors of crisp shapes

The estimation of the area of a 2D (crisp) object, and the volume of a 3D (crisp) object, from its digital image is usually done by counting pixels/voxels within the object. It is well-known that the relative error of these estimators is related to the sampling density (resolution) of an image, both in 2D and 3D case. The area (volume) estimation method, based on the grey-volume (the volume of a grey-level landscape) measurements, which combines counting pixels and edge detection, and exploits the sampling theory, is presented in Verbeek and van Vliet (1992).

The perimeter of a digital object can be estimated as the cumulative distance between neighbouring pixels on the border of an object. This method is based on the Freeman chain code representation (Freeman (1961)), where the weights 1 and $\sqrt{2}$ are assigned to isothetic and diagonal steps, respectively. These step weights are not optimal when measuring digitized straight lines. Optimized for infinitely long straight lines, the weights 0.948 and 1.343 are obtained for isothetic and diagonal steps, respectively, (Proffit and Rosen (1979), Kulpa (1977)). An overview of this type of estimators, which use local computations, is given by Dorst and Smeulders (1987). These weights are often used also for measuring the lengths of digital curves. Global estimators of perimeter are mostly based on a polygonalization of the object. A comparative evaluation of several estimators of the length of a binary digitized curve, both local and global, is presented by Coeurjolly and Klette (2004). The results show that length estimators which are multigrid convergent, i.e., perform better and better with increasing image resolution, are all global. However, both local and global methods produce either over- or under-estimation at low resolutions, i.e., when the object of interest is small in terms of the number of pixels.

In 3D space, the surface area of a digitized object can be estimated by counting the number of faces of the voxels at the boundary between the object and the background (Udupa (1994)). This gives a severe overestimation of the surface area. By approximating the boundary with a triangular representation, e.g., the one obtained from the Marching Cubes algorithm (Lorensen and Cline (1987)), more correct surface area estimates are obtained. An alternative approach is taken by Lindblad (2005), where surface area is estimated by using weighted local configurations, with weights optimized for planar surfaces. Multigrid convergent surface area estimators are studied in Coeurjolly et al. (2003); Kenmochi and Klette (2000).

In order to improve the precision of the estimations, methods based on grey-level images were developed. Estimation of the edge length (in 2D) and the surface area (in 3D) using grey-level information is analysed in Verbeek and van Vliet (1992); Eberly and Lancaster (1991); Eberly et al. (1991). Verbeek and van Vliet (1992) combine edge detection with the edge length (surface area) estimation technique, where edge length is expressed as a volume measure, and a so-called grey-volume estimator is then used. The methods presented in Eberly and Lancaster (1991); Eberly et al. (1991), for estimating arc length and 2D area, as well as surface area and volume, are based on unit normal vector construction at curve (surface) points. The normal construction is performed by using grey-level image data.

In the context of the perimeter and area estimators, it may be of interest to analyse how they affect the well-known P^2/A shape descriptor. A measure of the compactness of a shape S, also called *roundness index*, or *roundness factor*, is calculated as

$$P^2/A\,(S) = \frac{\text{perimeter}^2(S)}{4\pi \cdot \text{area}(S)}.$$

Based on the isoperimetric inequality,

$$\text{perimeter}^2(S) \geq 4\pi \cdot \text{area}(S), \tag{5}$$

the P^2/A measure is lowest – equal to one – for a crisp continuous disk, compared to any other crisp continuous set. In other words, the P^2/A measure is the lowest for the most compact continuous shape.

One possible extension of the 2D compactness measure to 3D is a $roundness$ measure of a shape S, calculated as

$$\text{roundness}(S) = \frac{\text{surface area(S)}}{\sqrt[3]{36\pi \cdot \text{volume}^2(S)}}\,. \tag{6}$$

In the continuous case, the $roundness$ measure of any object is larger or equal to one; it reaches its minimum for balls. In the discrete case, both the compactness and the roundness measures are under-estimated for digital disks and balls with small radii, so they can easily be lower than one.

4.4.2 Geometric descriptors of fuzzy shapes

The area (in 2D), and volume (in 3D), of a continuous fuzzy subset are defined as the integral of the corresponding membership function over the reference set. For the discrete case, it holds (Rosenfeld (1984)):

The area $A(S)$ (in 2D), and volume $V(S)$ (in 3D) of a (discrete) fuzzy subset S of a reference set X, given by its membership function μ_S, are defined as

$$\sum_{x \in X} \mu_S(x).$$

Note that area and volume are equal to the cardinality of a fuzzy set, according to one of several definitions of cardinality existing in the literature (Wygralak (1999)). Area and volume are scalar cardinalities, assigning a crisp number to a fuzzy set, as opposed to fuzzy cardinalities.

The perimeter $P(S)$ of a fuzzy set S given by a piecewise constant membership function (fuzzy step set) μ_S, is defined by Rosenfeld and Haber (1985) as

$$P(S) = \sum_{\substack{i,j,k \\ i<j}} |\mu_{S_i} - \mu_{S_j}| \cdot |A_{ijk}|,$$

where A_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

Even though the definitions given by Rosenfeld (1984) and Rosenfeld and Haber (1985) are generalisations of the area and perimeter of a crisp set, some simple inter-relations related to these measures, like e.g., the isoperimetric inequality, given by Relation (5), that hold in the crisp case, do not hold if the perimeter and area are defined as above. This fact initialized further research and resulted in the modified definition of the perimeter of a fuzzy subset, given in Bogomolny (1987):

The perimeter $P(S)$ of a fuzzy step subset S, given by its membership function μ_S, is defined as

$$P(S) = \sum_{\substack{i,j,k \\ i<j}} |\sqrt{\mu_{S_i}} - \sqrt{\mu_{S_j}}| \cdot |A_{ijk}|.$$

The idea followed by Bogomolny (1987) was to ensure that some well-known inter-relations between geometric properties of crisp objects, including the isoperimetric inequality, hold in a "proper" way for fuzzy objects as well. For that purpose, $\sqrt{\mu}$, instead of μ, is substituted in the definitions given previously in Rosenfeld and Haber (1985). Even though for a crisp set $\sqrt{\mu_S} = \mu_S$ (as $\mu_S \equiv 1$), and both definitions, Rosenfeld and Haber (1985) and Bogomolny (1987), provide, formally, the analogy with the definition of a perimeter of a crisp set, the difference between the two definitions is essential, and significantly affects the observed inter-relations in the two cases.

The behaviour of the compactness measure of fuzzy sets depends on the definition of the perimeter. It can be shown that the P^2/A measure decreases, i.e., that the compactness increases with the increase in fuzziness, if the perimeter is defined as in Rosenfeld and Haber (1985). A more intuitive result follows from the definition of perimeter given in Bogomolny (1987); the P^2/A measure indicates the crisp disk as the most compact fuzzy discrete shape.

In Papers V and VI, we studied perimeter and surface area, as well as area, volume, compactness and roundness, of discrete fuzzy objects, obtained by adjusting the definitions given in both Rosenfeld and Haber (1985) and Bogomolny (1987). Our main interest was to obtain good estimates of the analogous properties, of continuous crisp shapes. Statistical results show that fuzzy digital representations of a shape provide improved precision, compared to the results obtained if crisp digital representations are used. As expected, the improvements are most significant at low resolutions. Even though the approach presented in Bogomolny (1987) considers important principles, like inter-relations preservation, when generalizing notions from crisp to fuzzy sets, the corresponding measures give high over-estimates of continuous crisp analogues. We find that the approach suggested in Rosenfeld and Haber (1985) is more appropriate for estimations of the compactness measure of a real shape.

5 Contributions

This section contains a brief description of the results presented in Papers I–VII.

5.1 Moments of a spatial fuzzy set

The research presented in Papers I–III is focused on geometric moments and their application in representation and reconstruction of shapes.

General objects may require relatively large sets of moments for a reliable representation and reconstruction, which has the disadvantage of increased computational complexity, but even more important, increased sensitivity to noise due to the use of higher order moments. However, elementary geometric shapes, like disks and regular polygons, at least in the continuous case, are often uniquely represented by sets of moments of a relatively low order, while, in addition, such a description provides a unique reconstruction. These properties are appealing for a shape descriptor, which makes it interesting to study the behaviour of moments of discrete equivalents of regular shapes. Moreover, complex shapes most often can be decomposed into simpler shapes like ellipses (disks) or regular polygons, which makes the results obtained for simplified cases more generally applicable.

Our study on geometric moments of discrete regular shapes is performed on two main tracks:

- We aim to find moment-based descriptors for the observed set of discrete shapes such that a one-to-one correspondence between the shapes and their descriptors exists;
- We also aim to have an information on how well the moments of a continuous shape and the relevant shape parameters can be estimated from the moments of the discretization of a shape.

We are particularly interested in the effects of using a fuzzy shape representation on the accuracy and precision of the estimation of the relevant parameters of the continuous shape, by using moments.

The one-to-one correspondence between digital shapes and their moment descriptors is studied in Papers I and III. The shapes considered in Paper I are ellipses and disks in 2D, and ellipsoids and balls as their 3D extensions. Moments of their crisp digitizations of orders up to two are analysed. The study of disks is further extended to the case of fuzzy disks, and the results are presented in Paper III.

Error bounds for the estimation of moments of a continuous set are derived in Paper I, for crisp shapes, and in Papers II and III, for fuzzy shapes obtained by two different fuzzification approaches.

In order to explore the existence of a one-to-one correspondence between the set of discrete shapes and the set of their moments, we utilize the idea of separating sets, introduced in Klette et al. (1996). Let us observe two discrete sets, S_1 and S_2

in Z^2, both containing n points. Let us assume that there exists a curve $F(x, y) = 0$ such that

$$F(x, y) < 0 \text{ for all } (x, y) \in S_1 \quad \text{and} \quad F(x, y) > 0 \text{ for all } (x, y) \in S_2.$$

We say that F separates the sets S_1 and S_2.

If F is a linear combination of monomials $x^p y^q$, e.g., $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$, then, by summing n inequalities obtained for each point in the set S_1, it holds that

$$a \sum_{(x,y)\in S_1} x^2 + b \sum_{(x,y)\in S_1} xy + c \sum_{(x,y)\in S_1} y^2 + d \sum_{(x,y)\in S_1} x + e \sum_{(x,y)\in S_1} y + f \sum_{(x,y)\in S_1} 1 < 0$$

whereas, at the same time, for the set S_2 we get

$$a \sum_{(x,y)\in S_2} x^2 + b \sum_{(x,y)\in S_2} xy + c \sum_{(x,y)\in S_2} y^2 + d \sum_{(x,y)\in S_2} x + e \sum_{(x,y)\in S_2} y + f \sum_{(x,y)\in S_2} 1 > 0.$$

In other words,

$$\begin{aligned} & a\, m_{2,0}(S_1) + b\, m_{1,1}(S_1) + c\, m_{0,2}(S_1) + d\, m_{1,0}(S_1) + e\, m_{0,1}(S_1) + f\, m_{0,0}(S_1) \\ & < a\, m_{2,0}(S_2) + b\, m_{1,1}(S_2) + c\, m_{0,2}(S_2) + d\, m_{1,0}(S_2) + e\, m_{0,1}(S_2) + f\, m_{0,0}(S_2). \end{aligned} \tag{7}$$

It is easy to conclude that S_1 and S_2 cannot have all of the corresponding moments, appearing in inequality (7), equal, which indicates that an appropriate choice of descriptor for these shapes can be $(m_{2,0}; m_{1,1}; m_{0,2}; m_{1,0}; m_{0,1}; m_{0,0})$. Clearly, the form of F determines which moments to choose for the description.

Note that the linearity of moments allows us to apply the idea of separation even when S_1 and S_2 have a non-empty intersection. In that case, a curve that separates $S_1 \setminus S_2$ and $S_2 \setminus S_1$ is used, taking into account that the contributions of the points in $S_1 \cap S_2$ to the moments of S_1 and S_2 are equal.

In Paper I, this idea is applied to crisp discretizations of ellipses, disks, ellipsoids, and balls, and the choice of moments, providing a one-to-one correspondence between the set of discrete shapes and the set of moment descriptors, is made in each case, depending on the separating curve.

As an example, let us observe that the difference sets of two isothetic digital ellipses can always be separated by a curve $F(x, y) = ax^2 + bx + cy + d$, as illustrated in Figure 10. Therefore, the descriptor $(m_{2,0}; m_{1,0}; m_{0,1}; m_{0,0})$ provides a unique representation of this type of shapes.

A representation (coding) of ellipses by moments requires an asymptotically optimal amount of bits. It provides recoverability of the shape from its code, due to the one-to-one correspondence established. In addition, this coding scheme is based on integers, which implies that the floating point calculation errors and approximations unavoidably introduced if real numbers are used, are avoided.

In Paper III, digital fuzzy disks are interpreted as 3D crisp shapes, in order to apply the same separation technique. This initiated the introduction of the generalized

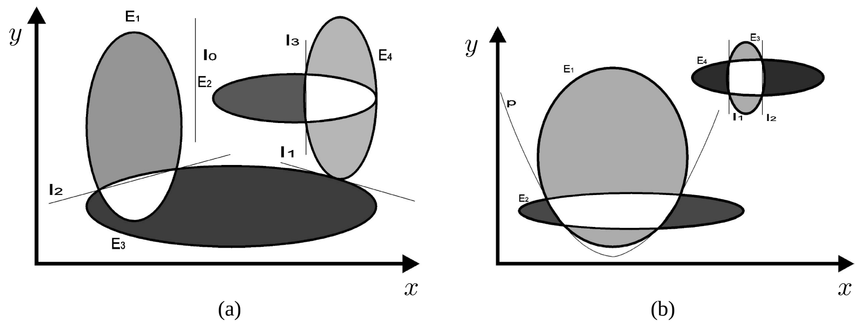


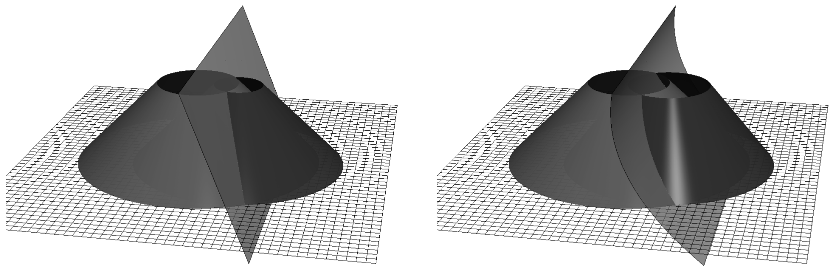
Figure 10: Separating curve for the difference sets of two isothetic ellipses. (a) A separator is of the form $ax+by+c=0$ if two ellipses have at most three intersecting points. For the difference sets of ellipses E_1 and E_2, E_3 and E_4, E_1 and E_3, and E_2 and E_4, having zero, one, two, and three intersecting points, respectively, separators are straight lines l_0, l_1, l_2, and l_3, respectively. (b) A separator is of the form $ax^2+dx+ey+f=0$ or $ax^2+dx+f=0$ if two ellipses have four intersecting points. The difference sets of ellipses E_1 and E_2 are separated by parabola p, while the difference sets of E_3 and E_4 are separated by two parallel straight lines, l_1 and l_2.

moments of a fuzzy shape; it proved to be necessary to incorporate membership direction information into the description of a 2D fuzzy shape. The membership direction can be interpreted as an additional (third) spatial dimension. A guidance regarding which moments to introduce, and the proof of uniqueness of representation were provided by the existence and the form of a separating surface. Figure 11 presents these surfaces.

Accuracy and precision of estimation of moments of a continuous shape from the moments of its digitization is studied in Papers I–III. Knowing that both the accuracy of the estimation and the computational complexity increase with the increased resolution of an image, we were interested in answering the following questions:

- What is the spatial image resolution that should be used in order to obtain the required precision in the reconstruction of the original image?
- If we are not able to increase the spatial resolution of an image in order to achieve an increased accuracy of the shape moment estimation (which is often the case for real images), how much do we gain if we use a fuzzy representation of the object instead of a crisp one?

The dependence of the estimates from the image resolution is conveniently studied by using a multigrid approach. An object is inscribed into the grid and subsequently either dilated r times, or equivalently, observed in an r times refined grid, for increasing values of r.



(a) (b)

Figure 11: 3D illustrations of two intersecting fuzzy disks (light and dark grey) having (a) the same and (b) different border widths. The disks in (a) are separated by a plane, whereas to separate the disks in (b) a parabolic surface is required.

All the results are based on the strong number-theoretical results derived by Huxley (1990), related to the estimation of the the number of grid points inside of a 3-smooth convex set. In Paper I, Huxley's result is incorporated in the estimates of moments up to order two, for isothetic ellipses. Either by generalization, or by reduction to a special case, estimates are derived for ellipses in general position, ellipsoids, disks, and balls. They are further generalized by Klette and Žunić (2000) to any convex 2D shape.

Being interested in fuzzy approaches, we performed a study of possible further generalization of the error bounds given by Klette and Žunić (2000) to the estimations from shapes fuzzified by area coverage approach. The results from Paper II prove that it is possible to overcome problems of insufficient available spatial resolution by utilizing an often already existing membership resolution. Derived asymptotic expressions for the estimation of moments of convex shapes from their fuzzy discrete representations show that an increase in membership resolution of an image can be used to achieve the improved accuracy of the estimation in the same way as by an increase in the spatial resolution. Even though the theory only guarantees this behaviour after a certain spatial resolution is reached, the simulations show that it is also present at low resolutions.

The asymptotic expressions are derived by observing the correspondence between the supersampling of (border) pixels in order to derive fuzzy memberships in accordance with the area coverage and the refinement in a crisp multigrid approach. We notice that the difference between local contributions obtained from the two approaches increases with an increase in the order of a moment, due to the increased influence of the pixel position to the moment calculation. As an example, we can observe one fuzzy pixel located at position (x, y) on the border of the imaged object, in an image with a spatial resolution r_s and a membership resolution 16, or equivalently, a block of 4×4 crisp pixels in an image with a spatial resolution $r_s \times r_f$, as presented in Figure 12.

The presented configuration is described by the number $k = 5$ of (crisp) sub-

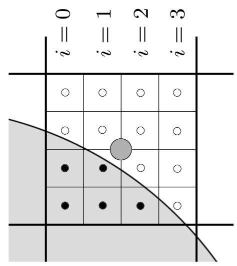


Figure 12: A pixel on the fuzzy border of a discretized object and, equivalently, a 4×4 block of crisp pixels with a 4 times dilated object.

pixels with centroids covered by the object,the super-sampling factor $r_f = 4$, and the positions (i, j) of sub-pixels within the 4×4 block. The local contribution of this configuration to the moments up to the order two is calculated as:

$$\begin{aligned}
c\tilde{m}^{(1)}_{0,0}(r_s r_f S) &= k = 5 \ , \\
f\tilde{m}^{(1)}_{0,0}(r_s S) &= \frac{k}{r_f^2} = \frac{5}{16} \ ,
\end{aligned}$$

$$\begin{aligned}
c\tilde{m}^{(1)}_{1,0}(r_s r_f S) &= k\left(r_s r_f x - \frac{r_f}{2} + \frac{1}{2}\right) + \sum i = 20 r_s x - \frac{7}{2} \ , \\
f\tilde{m}^{(1)}_{1,0}(r_s S) &= r_s x \ \frac{k}{r_f^2} = \frac{5}{16}\, r_s x \ ,
\end{aligned}$$

$$\begin{aligned}
c\tilde{m}^{(1)}_{2,0}(r_s r_f S) &= k\left(r_s r_f x - \frac{r_f}{2} + \frac{1}{2}\right)^2 + 2\left(r_s r_f x - \frac{r_f}{2} + \frac{1}{2}\right)\sum i + \sum i^2 \\
&= 90 r_s^2 x^2 - 28 r_s x + \frac{21}{4} \ , \\
f\tilde{m}^{(1)}_{2,0}(r_s S) &= r_s^2 x^2 \ \frac{k}{r_f^2} = \frac{5}{16}\, r_s^2 x^2 .
\end{aligned}$$

The notation is the same as in Paper II; $c\tilde{m}_{p,q}$ corresponds to the (p, q)-moment of a crisp discrete shape, while $f\tilde{m}_{p,q}$ denotes the (p, q)-moment of a fuzzy discrete shape. Upper index (1) stands for a one pixel contribution.

The above calculations agree with the theoretical expressions for one fuzzy pixel/crisp block contributions, derived for the worst case when a half of the sub-pixels are included in the digitization:

$$\begin{aligned}
c\tilde{m}^{(1)}_{0,0}(r_s r_f S) &= r_f^2 \ f\tilde{m}^{(1)}_{0,0}(r_s S) \\
c\tilde{m}^{(1)}_{1,0}(r_s r_f S) &= r_f^3 \ f\tilde{m}^{(1)}_{1,0}(r_s S) + \mathcal{O}\left(r_f^3\right) \\
c\tilde{m}^{(1)}_{2,0}(r_s r_f S) &= r_f^4 \ f\tilde{m}^{(1)}_{1,0}(r_s S) + \mathcal{O}\left(r_s r_f^4\right) .
\end{aligned}$$

Assuming that the worst case configurations can appear only on the border of the object, while deriving that the inner (fully covered) pixels do not introduce any error, we obtain that the order of error of the estimation of moments up to the order two of the continuous convex set is reduced by a factor r, when either spatial resolution, or membership resolution, is increased by the factor r. (Note that in the worst theoretical case, where all the pixels in the image are half covered by the object, the order of estimation error is the same as in the crisp case.)

Our statistical study shows high accordance with the theoretical results. Tests are performed for 2 000 randomly positioned disks and for 10 000 randomly positioned squares, for each of the observed increasing sizes of the objects. This approach is used to express multigrid resolution. Squares are of interest as convex objects containing straight line borders, while disks are studied as examples of 3-smooth objects. The different membership resolutions used are expressed by observing $r_f \in \{1, 2, 4, 8, 16\}$.

For each object size, we determine the maximal relative estimation error for moments up to order two. The results for $m_{2,0}$ moments estimation, both for squares and for disks, are shown in Figure 13. The plots are presented in a logarithmic scale so that the "slope" of the curves corresponds to the order of the estimation error, and can be compared with the straight line having the slope equal to the theoretically derived order of error, which is plotted, as well. The expected asymptotic behaviour assumes higher errors at low resolutions. However, for the presented membership resolutions, the plots show accordance with the asymptotic bounds even at low spatial resolutions. The relative position of the curves shows that the estimation error becomes smaller both with the increase in spatial and membership resolution.

Fuzzification by area coverage, considered in Paper II, is of interest because of its good correspondence with the output of many imaging devices used in practise. It does not, however, preserve all of the shape properties, which makes theoretical studies related to, e.g., uniqueness and recoverability of the representation by moments difficult. For example, according to the definition given by Bogomolny (1987), a fuzzy disk is a convex fuzzy set defined by a membership function which depends only on the distance of a point from the centre of a disk. A pixel area fuzzification of a disk does not necessarily produce a fuzzy discrete disk, as shown in Figure 14; for the indicated pixels A and B it holds that $d_A = d_B$, while $\mu(A) < \mu(B)$.

In Paper III, we studied properties of moments of a theoretically well defined continuous fuzzy object, as well as of its digitization. By introducing the notion of generalized moments of a fuzzy set, we have proved a one-to-one correspondence between the set of fuzzy discrete disks and the set of their moments. Generalized moments, inspired by the possibility to interpret 2D fuzzy sets as 3D crisp objects, enable the use of already developed techniques, while a different treatment of spatial and membership information can, but need not, exist.

We have studied the accuracy of the estimation of moments of a continuous fuzzy disk from the moments of its digitization, and have derived error bounds for the estimates. It is shown that the error depends both on spatial and membership resolution. Moreover, it is concluded that increasing only one type of resolution can provide only a limited improvement of the accuracy of estimates; in other words,

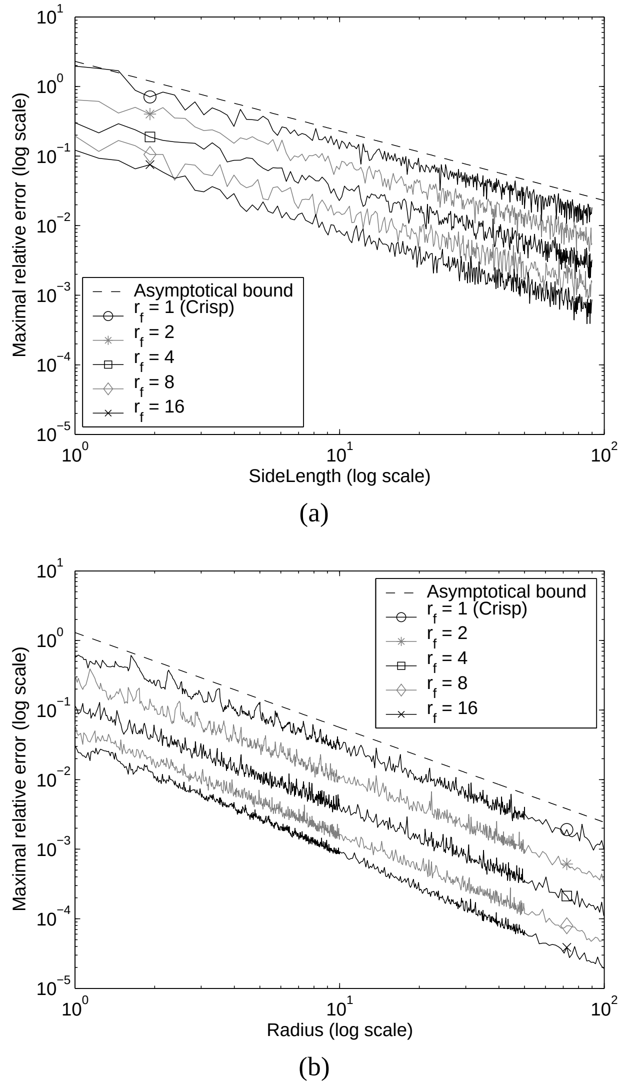


Figure 13: Error bounds for the second order moment estimation. (a) Moments estimation of a square. (b) Moments estimation of a disk.

spatial and membership resolution should be well balanced in order to get an optimal estimation result. The results are presented in Figure 15, where maximal relative errors (in logarithmic scale) for (generalized) moments of the first order are plotted. The lines corresponding to the asymptotes of the theoretical expressions are included in the plots, to simplify the comparison with the theoretically derived error bounds.

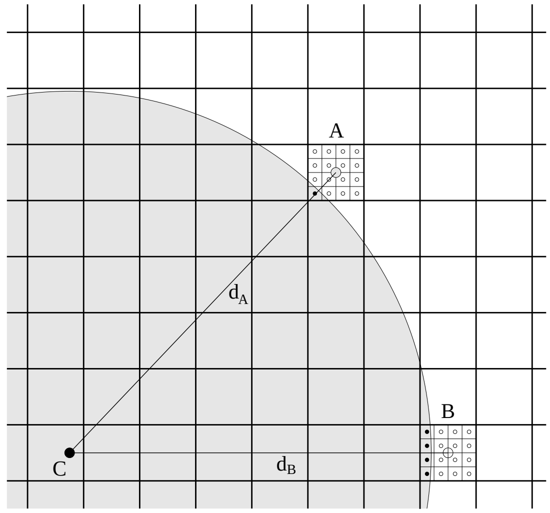
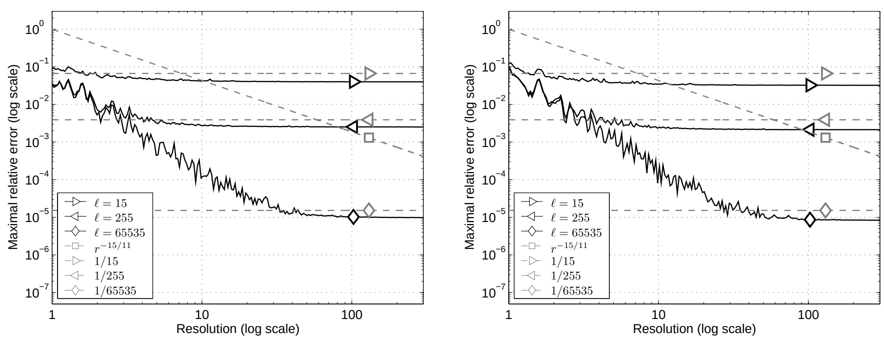


Figure 14: Pixel area coverage fuzzification of a disk does not necessarily produce a fuzzy discrete disk.



(a) (b)

Figure 15: Maximal relative errors for the estimation of moments of a continuous fuzzy disk from the moments of its digitization at different resolutions. (a) $m_{1,0;0}$ (b) $m_{0,0;1}$

We conclude by listing three theorems expressing error bounds for moment estimations. The first one is related to estimation of moments of a crisp continuous convex set from the moments of its crisp digitization, (Klette and Žunić (2000)), the second one is related to the estimation of moments of a crisp continuous convex set from the moments of its fuzzy digitization (Paper II), whereas the third theorem is related to the estimation of moments of a fuzzy continuous disk from the generalized moments of its fuzzy digitization (Paper III).

Theorem 1 Klette and Žunić (2000) *The moments $cm_{p,q}$ of a crisp convex set S, with a boundary consisting of a finite number of C^3 arcs, can be estimated from the digitization $\mathcal{D}(r_s S)$ of an r_s times dilated set S, by the estimation formula*

$$cm_{p,q}(S) = \frac{1}{r_s^{p+q+2}}\ c\tilde{m}_{p,q}(\mathcal{D}(r_s S)) + \mathcal{O}\left(\frac{1}{r_s}\right)$$

for $p+q \leq 2$. If S is a 3-smooth set, the estimation formula is

$$cm_{p,q}(S) = \frac{1}{r_s^{p+q+2}}\ c\tilde{m}_{p,q}(\mathcal{D}(r_s S)) + \mathcal{O}\left(\frac{1}{r_s^{\frac{15}{11}-\varepsilon}}\right)$$

for $p+q \leq 2$.

Theorem 2 (Paper II) *The moments of a convex set S, with a boundary consisting of a finite number of C^3 arcs, can be estimated from the digitization of $r_s S$, fuzzified by area coverage fuzzification in membership resolution r_f^2, by the estimation formula*

$$cm_{p,q}(S) = \frac{1}{r_s^{p+q+2}}\ f\tilde{m}_{p,q}(\mathcal{D}(r_s S)) + \mathcal{O}\left(\frac{1}{r_s r_f}\right), \quad \text{for } r_s > C r_f,$$

for $p+q \leq 2$. If S is a 3-smooth set, the estimation formula is

$$cm_{p,q}(S) = \frac{1}{r_s^{p+q+2}}\ f\tilde{m}_{p,q}(\mathcal{D}(r_s S)) + \mathcal{O}\left(\frac{1}{(r_s r_f)^{\frac{15}{11}-\varepsilon}}\right), \quad \text{for } r_s > C {r_f}^{\frac{15}{7}+\varepsilon},$$

for $p+q \leq 2$. C is a constant.

Theorem 3 (Paper III) *The generalized moments of a continuous convex fuzzy disk S with bounded support, can be estimated from the generalized moments of it digitization in a grid with spatial resolution r_s and membership resolution ℓ, by the estimation formula*

$$fm_{p,q;\phi}(S) = \frac{1}{{r_s}^{p+q+2}\ \ell^{\phi+1}}\ f\tilde{m}_{p,q;\phi}(\mathcal{D}_\ell(r_s S)) + \mathcal{O}\left(\frac{1}{r_s^{\frac{15}{11}-\varepsilon}}\right) + \mathcal{O}\left(\frac{1}{\ell}\right),$$

for $p+q \leq 2$ and $\phi \leq 1$.

5.2 Shape signature of a spatial fuzzy set

In the continuous case, a shape signature based on the distance of the boundary points from the centroid of the shape is translation-invariant. However, the discretization of a shape in general depends on the position of the shape within the grid, which affects many properties of discrete objects. With intention of increasing the invariance of the signature of a discrete shape under translation, in Paper IV we suggest to represent a shape by a fuzzy discrete spatial set, and to use the signature of a fuzzy shape as a descriptor.

Even though it may seem more appropriate to use region-based shape descriptors to describe fuzzy shapes, since possible membership variations can be captured if the properties of the whole region are considered, there are reasons to consider the adjustment of boundary-based approaches to fuzzy sets. Boundary-based shape descriptors are often very sensitive to noise and to any change of the boundary position. Translation of the shape centroid within a pixel can affect crisp segmentation, and consequently the border position, quite significantly, which further propagates to the shape signature. On the other hand, a fuzzy border is often less sensitive to the translation of a shape, so it seems natural to use a fuzzy representation for the shape description. Such descriptions might be particularly appropriate when fuzziness appears primarily on the border of the object, which is often the case in image analysis.

Another reason to believe that the fuzzy set representation can be appropriate for boundary-shape description is because it is often possible to incorporate information about the inner region of the shape into the descriptor, even if the main focus is on the boundary points. Such additional information can improve the reliability of the description.

In Paper IV, we analysed how to define and calculate the radius-vector function of a fuzzy set. The extension is derived both for the continuous and the discrete cases, and a significant difference between them is noticed, first of all, regarding the uniqueness of the solution.

The centroid (centre of gravity) of a continuous fuzzy set is defined by Equation (4). Analogously to the crisp case, the applicability of the signature function is restricted to fuzzy sets which are star-shaped with respect to the centroid. A fuzzy set is star-shaped with respect to a point O if all its α-cuts are star-shaped with respect to O (Diamond (1990)). This implies that all the α-cuts of a fuzzy set, which is star-shaped with respect to the centroid, contain the centroid. It also follows that the centroid of such a set has maximal membership to the fuzzy set.

Depending on the representation of a fuzzy set, (1) by its membership function, or (2) as the stack of its α-cuts, two approaches for defining the signature of a fuzzy shape are suggested. They differ in the choice of a distance measure and in the interpretation of the boundary of a fuzzy set.

The first approach relies on the representation of a fuzzy set by its membership function. For a fuzzy star-shaped set, the membership function is radially non-increasing and, for a set with a bounded support (which is an assumed property of the fuzzy sets we observe), it is equal to zero outside the support of the fuzzy set. The signature of a fuzzy set is calculated by radial integration of a membership function from the centroid of a shape. The contribution of the region outside the support of the set to the integral is equal to zero. Therefore, the signature is determined by the integration performed from the centroid of the fuzzy set to the boundary of its support. The length of a path in a fuzzy set is defined as the integral of the membership function along the path, as suggested in Saha et al. (2002). If applied to a crisp set, this method reduces to the conventional signature of the set.

The second suggested approach is based on the representation of a fuzzy set by the stack of its α-cuts. To obtain the signature function of a fuzzy set, the fuzzification principle given by Equation (1) is applied. That means that the signature of a fuzzy set is calculated (by integration) from the signatures of all the α-cuts of a fuzzy set.

The two suggested methods are equivalent in the continuous case, which is a consequence of the properties of the membership function of a fuzzy star-shaped set, being radially non-increasing. One radial cut of a membership function corresponds to the path to be measured in order to determine the distance to the boundary point, but the same radial cut can also be seen as the signature-profile corresponding to the boundary point in the same radial direction for all the α-cuts. Integration in both cases gives the same result, assigned as the signature value to the observed point. However, the issues induced by the discretization lead to different performances of the proposed methods, when they are applied to discrete shapes.

For both suggested methods, the centre of gravity (centroid) of a discrete set is computed by taking the sum, instead of the integral, in Equation (4). The boundary of the support of a discrete fuzzy set (Method 1), or the boundaries of its α-cuts (Method 2), are extracted. The number of observed α-cuts is finite and is equal to the number of grey-levels representing different membership values of the points.

For Method 1, the straight continuous line segments between the centroid and each of the points on the discrete boundary are discretized and the length of a discrete path in a fuzzy set is determined as suggested in Saha et al. (2002), i.e., approximating the integral by a finite integral sum along the discrete line segment.

For Method 2, the signature of a fuzzy set is obtained by averaging the signature functions calculated for each of the α-cuts. The signature of an α-cut is determined by using the Euclidean distance to each of the points on the discrete boundary. Since the boundaries of the α-cuts in general do not contain the same number of points, resampling of the signatures is needed, in order to obtain all of them having the same length (equal to the longest). Care is taken so that the correspondence between the points is preserved, and indexing is such that the same index in all of the signatures indicates the border points belonging (approximately) to the same radial cut.

The results obtained for a digital disk are presented in Figure 16. Let us recall that the signature of a continuous disk is a constant function which maps each boundary point to the value equal to the radius of the disk.

The performances of the methods are studied on a set of digital fuzzy disks, positioned randomly within a pixel. Fuzzification based on area coverage is used. The evaluation is based on signal-to-noise ratio (SNR), which is calculated as the ratio between the total of squared values of the observed signal and the total of squared values of the noise contribution.

The average SNR values obtained for 50 disks of each observed real-valued radius are presented in Figure 17, for increasing disk radii. The SNR is calculated in a way that emphasizes the precision of the methods, rather than their accuracy (the mean of estimates is used instead of the truth value). A motivation is that the most significant improvement obtained by using fuzzy, instead of the crisp shape representation, is expected in terms of the precision of the estimates.

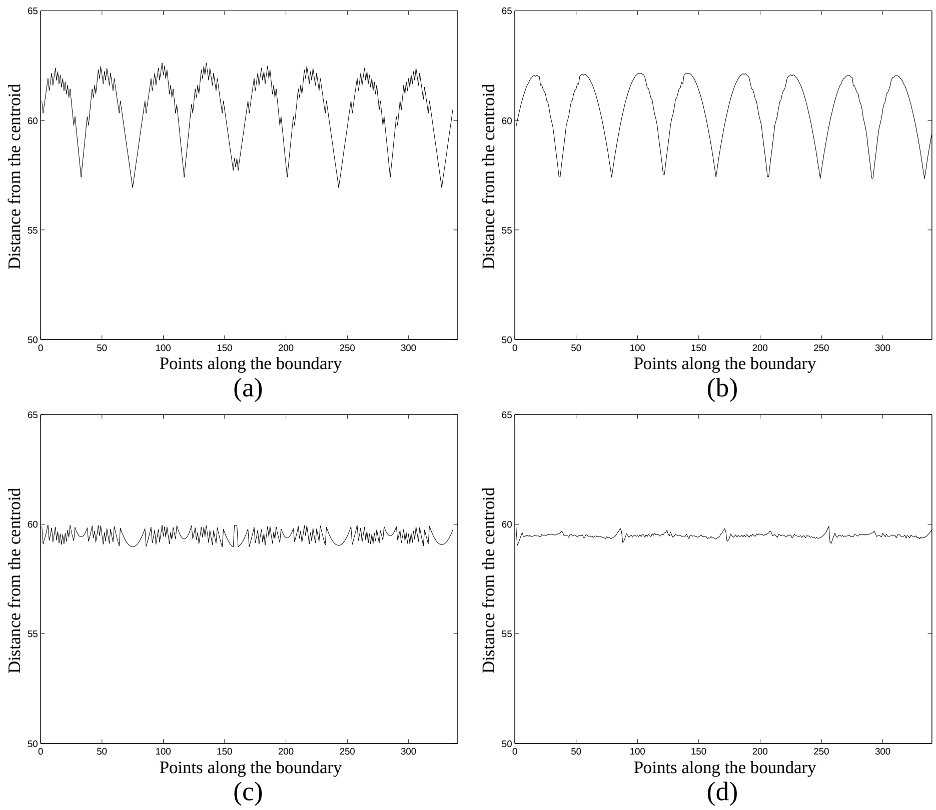


Figure 16: Shape signatures for a disk. Method 1 applied to (a) a crisp and (b) a fuzzy disk. Method 2 applied to (c) a crisp and (d) a fuzzy disk.

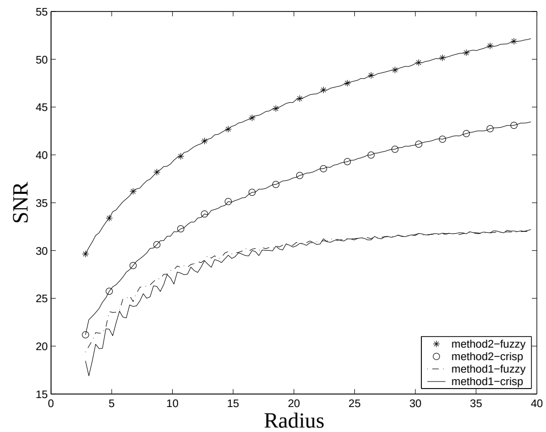


Figure 17: SNR of computed shape signatures for disks; comparative study of the two methods on crisp and fuzzy shapes.

It can be noticed that for both methods, the use of a fuzzy, instead of a crisp object, improves the description. However, for Method 1, the improvement tends to zero when the radius increases. Method 2 greatly outperforms Method 1, both in the crisp and in the fuzzy case. Furthermore, for Method 2 the advantage of using fuzzy objects is obvious and remains so also with the increase in the radius of the object.

The poor performance of Method 1 is not surprising, since the method relies on a discrete approximation of a straight line, and on the estimated length of a line segment, while Method 2 directly uses Euclidean distances. However, the first method can be seen as more general, since it can naturally be extended to shape signature calculation of non-star-shaped sets, while non-star-shapedness causes problems (also in the crisp case) when the second approach is used. The difficulties are related to the treatment of "external" parts of the line connecting the centroid with the boundary point, where "external" becomes a rather subtle notion for the relatively complex fuzzy topological issues. Hence, this is a reason to be interested in the less efficient approach.

An important conclusion is that the use of fuzzy sets greatly improves the quality of the shape description; the sensitivity to the digitization effects is significantly reduced, compared to the crisp case. An example presented in Figure 18, where signatures of a rectangle in two rotations are compared, while the same starting boundary point is used in both positions. The resulting signatures should, theoretically, be the same. Due to discretization, they are not, in neither the crisp, nor the fuzzy case. However, fuzzy representation of an object provides a more stable description.

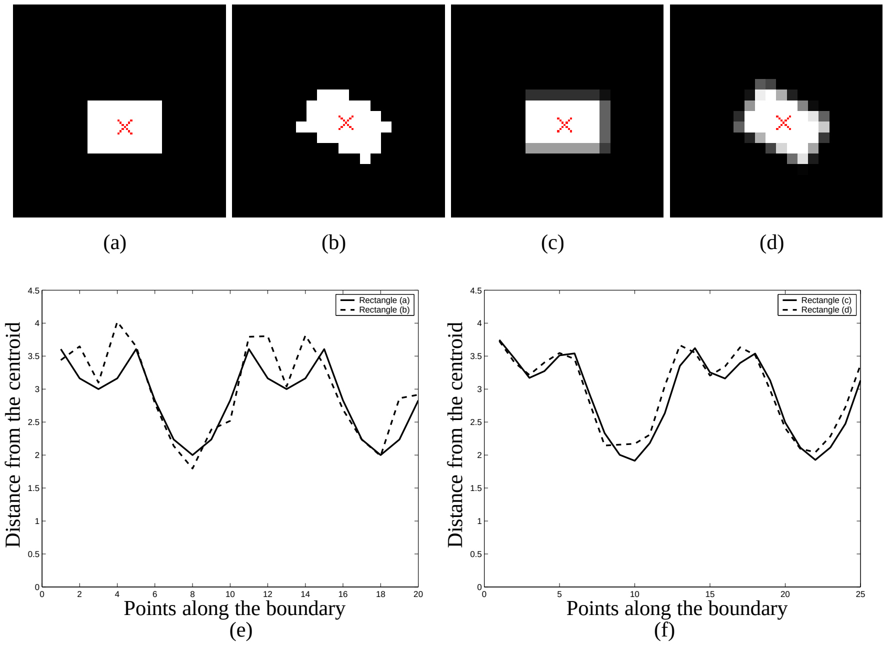


Figure 18: Four digitizations of a rectangle and the corresponding signatures. Centres of gravity are marked by crosses. The boundaries are traversed in the counterclockwise direction, starting from the upper left pixel of the support. (a) Crisp isothetic. (b) Crisp rotated. (c) Fuzzy isothetic. (d) Fuzzy rotated. (e) Signatures of the shapes in (a) and (b). (f) Signatures of the shapes in (c) and (d).

5.3 Geometric descriptors of a spatial fuzzy set

In Papers V and VI we studied the behaviour of the perimeter and the area of a discrete fuzzy shape, in order to use them in estimations of the perimeter and area of continuous crisp shapes. We extended the study to the 3D case, i.e., to the estimations of the surface area and the volume of an object. We utilized already existing definitions of the area and perimeter of a fuzzy set, given by Rosenfeld and Haber (1985) and Bogomolny (1987). Our goals were to

- appropriately adjust the existing definitions to a discrete case;
- propose an algorithm for calculating the perimeter of a discrete fuzzy set;
- use the obtained results for estimations of measures of continuous crisp shapes and perform a study on the precision of the designed estimators, with focus on small objects;
- extend the obtained results to 3D objects.

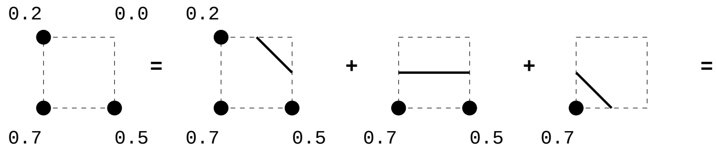
As we expected, one of the advantages of a fuzzy approach was shown to be in the increased stability of the fuzzy representation of a continuous shape under digitization, compared to a crisp representation, present especially at low spatial resolutions (i.e., small objects). For an illustration, the objects in Figure 18(b) and (d) can be compared; they are both digitizations of the same continuous rectangle, and the shape properties are far better preserved in (d), where a fuzzy representation (based on area coverage) is used, than in (b), where a crisp digitization is shown. Another important advantage of a fuzzy approach in the perimeter (and area) estimation method presented in Papers V and VI is that there is no need for (crisp) boundary detection; the estimation is done directly on the fuzzy segmented discrete object.

It is reasonable to expect that the area of a fuzzy set, based on area coverage fuzzification, provides a good estimate of the area of a corresponding crisp continuous original. Memberships of pixels are derived so that the contribution of a pixel to the area of an object is reflected. Summing memberships in order to estimate the area gives a very precise result, which is supported by a statistical study performed for a set of disks and a set of squares of various sizes, randomly positioned in the digitization grid. At this point, we mention that the theoretical error bounds for the estimation of the moment of zero-order, derived in Paper III, confirm increased precision of the area estimator based on fuzzy memberships of the points.

Similarly, the volume of a 3D continuous shape is estimated with high precision by summing memberships of voxels to the corresponding fuzzy digital shape, i.e., by the volume of the corresponding fuzzy digital shape. The results of our statistical study, performed for small balls, are presented in Paper VI.

In order to design an algorithm for computation of the perimeter of a fuzzy set, we considered the definitions given by Rosenfeld and Haber (1985) and Bogomolny (1987), which lead to computation based on local contributions of pixels to the perimeter, and also to the fuzzification principle expressed by Expression (1), which assumes computation of the perimeter of the fuzzy set as an averaged sum of perimeters of all the α-cuts of the set. Algorithms based on local contributions are given in both Papers V and VI. Since both definitions of a perimeter, given by Rosenfeld and Haber (1985) and Bogomolny (1987), require the calculation of the length of a border line between two sets of iso-membership pixels, the accuracy of the estimation of a perimeter of a continuous set from its fuzzy digital representation strongly depends on the accuracy of the estimation of the length of a crisp digital curve.

The algorithm which incorporates a length estimation method given in Dorst and Smeulders (1987) into the computation of the perimeter of a fuzzy set is presented in Paper V. The algorithm is restricted to locally convex shapes (i.e., sets where the 3×3 neighbourhood of each pixel is a convex set), and based on weighted local (isothetic and diagonal) steps. It is implemented for both considered perimeter definitions, and tested on a set of disks and a set of squares. Estimations exhibit improved precision, compared to the crisp digital case; when observed for small digital disks, with radii up to 20 pixels, estimates based on the definition given by Rosenfeld and Haber (1985) are more accurate than those based on the definition of Bogomolny (1987).



$$= \frac{b}{2}(0.2-0.0) + a(0.5-0.2) + \frac{b}{2}(0.7-0.5)$$

Figure 19: Calculation of the contribution to perimeter of a fuzzy m-square. There is a pixel in each corner of the square. Black dots are set pixels.

In order to further improve perimeter estimation based on fuzzy sets, and to avoid restriction to locally convex shapes, in Paper VI we present an algorithm where we utilize a marching squares method, as an alternative length estimator. Such an approach provides increased accuracy of the estimates, compared to the method used in Paper V, since it incorporates a more accurate estimation of the perimeter of the crisp set. In addition, it can be applied to any fuzzy shape, not only to locally convex ones.

A marching square (m-square) is the square bounded by the centres of 4 pixels in a 2×2 neighbourhood. If object and background are considered (i.e., if the configuration is binary) there are $2^4 = 16$ possible configurations of pixels forming an m-square. Grouping symmetry and complementary cases results in 4 different configurations. The length of the border, corresponding to these configurations, is equal to

- 0, when all 4 pixels are set or when none of them is set;
- $\frac{b}{2}$, when one of the 4 pixels is set or when one of them is not set;
- a, when two edge-neighbours are set;
- b, when two vertex-neighbours are set,

where a- and b-step are the estimates (weights) of the isothetic and the diagonal distance between two neighbouring pixels, respectively. We use $a_{MSE_{n\to\infty}} \approx 0.948$ and $b_{MSE_{n\to\infty}} \approx 1.343$ (Dorst and Smeulders (1987), Kulpa (1977)) to minimize the expected mean square error (MSE) for measurement of the length of long line segments (when length tends to infinity) and to give an unbiased estimate.

The perimeter of a digital fuzzy subset is obtained by summing the contributions for each 2×2 configuration. The individual contribution is calculated by summing the differences between membership values of the pixels belonging to two successive α-cuts in the m-square multiplied by the lengths of the corresponding border lines. An example of the calculation of a local contribution of a fuzzy Marching Square is shown in Figure 19.

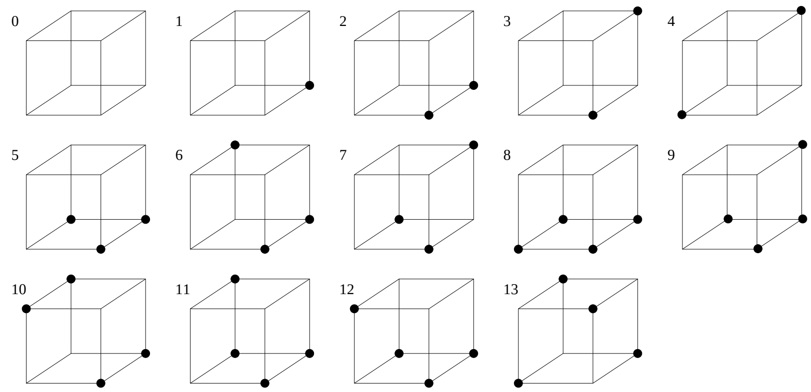


Figure 20: Marching cubes of $2 \times 2 \times 2$ voxels. Voxels denoted by $\bullet$ are inside the object.

Note that the suggested local calculation provides the same global result as the weighted summation of the perimeters of all the α-cuts, which corresponds to the fuzzification principle expressed by Expression (1). This follows from the fact that for each 2×2 configuration a corresponding part of the (crisp) boundary, that would occur at each α-cut, is considered; a part is weighted by the difference to the closest higher α-cut. After scanning the whole image, all crisp boundaries, appearing at each α-cut, are generated, with the assigned corresponding weights. However, the summation is done "vertically", instead of "horizontally". A similar result is obtained in Dubois and Jaulent (1987), where the expected perimeter is shown to be in accordance with Rosenfeld and Haber (1985). An important property of our implementation is that local calculations are sufficient. Neither generation of α-cuts, nor detection of their borders, are necessary. The complexity of the algorithm is a linear function of the number of image points.

Paper VI extends the proposed perimeter estimation method to 3D. A surface area estimator, incorporating Marching Cube configurations (Lorensen and Cline (1987)), presented in Figure 20, and local surface contribution weights derived by Lindblad (2005) for crisp digital surface area estimation, is developed and its performance for digital balls with fuzzy borders is analysed. The approach taken in Rosenfeld and Haber (1985) is generalized, since it shows higher precision of estimates in 2D, compared to the approach of Bogomolny (1987). The results show high accuracy and precision of the estimates, compared to existing binary surface area estimation results (Coeurjolly et al. (2003); Kenmochi and Klette (2000)), especially in the case of small digital objects, having up to 20 voxels in diameter.

In both Paper V and Paper VI, the measures of compactness, P^2/A in 2D and $roundness$ in 3D, are analysed. We conclude that the estimate for P^2/A measure for the disks is rather close to the true value (that is 1 for real disks). Not only is the precision of the estimation improved by introducing fuzzy segmentation, but the P^2/A measure is stabilized to be larger than 1. In other words, the isoperimetric inequality is satisfied at lower spatial resolution than in the crisp case. In the 3D

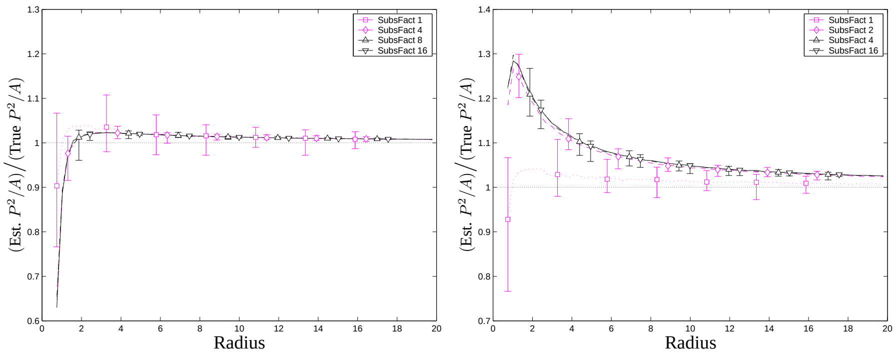


Figure 21: P^2/A measure estimation for digitized disks. Left: Results based on the definitions by Rosenfeld and Haber (1985). Right: Results based on the definitions by Bogomolny (1987) .

continuous case, the $roundness$ measure of any object is larger or equal to 1; it reaches its minimum for continuous balls. In the discrete case, however, this measure is under-estimated for digital balls with small radii. Using the obtained estimations for the surface area and volume of a fuzzy digital shape, we get a roundness measure which is still under-estimated, but the precision of the results is improved.

The perimeter definition suggested in Bogomolny (1987), when adjusted to discrete shapes, leads to the P^2/A measure which indicates the crisp discrete disk as the most compact fuzzy discrete shape. Moreover, the consequences of the definition of Bogomolny (1987) propagate to the compactness measure not only by ensuring that it has a value higher than one, but also by providing that it has a more intuitive behaviour; increase in fuzziness leads to decrease in compactness. However, the P^2/A with definition of Bogomolny (1987) gives high over-estimates when used for estimation of P^2/A of a crisp continuous object, and therefore it is less appropriate for approximations of real shape compactness measure. Statistical results obtained for digital disks are shown in Figure 21. The different curves denote different levels of fuzziness.

According to the results presented in Papers V and VI, the perimeter estimation method based on the definition of Rosenfeld and Haber (1985) considerably improves estimation precision even in the very delicate case of a shape bounded by straight line segments aligned with the grid. That is the reason we suggest to select Rosenfeld's definition, whenever the precision of the estimates is a priority. We believe, however, that Bogomolny's definition treats some important geometric properties of fuzzy sets in a good and natural way, and is worth further exploration.

5.4 Defuzzification based on feature invariance

In Papers VII and III, we are interested in generating a crisp representation of a fuzzy digital object. Such a process is known as defuzzification, and can be performed either as an inverse of fuzzification, with the intention to recover the fuzzified crisp original, or as a process independent of any fuzzification, but based on some predefined conditions that should be fulfilled for a crisp set to be the representation of a given fuzzy set. We conclude that the applications in image analysis require a combination of the two views. The goal is to reconstruct the crisp continuous original from its fuzzy digital representation, but the fuzzification function is rarely known, and practically never analytically defined; consequently its inverse cannot be used. Instead, a fuzzy representation of a set is used as a source of valuable information about the geometric properties of the object that was fuzzified.

In Paper VII, we present a defuzzification method which generates a crisp object having area, perimeter, and centre of gravity as close as possible to the corresponding features of the fuzzy set, while preserving the similarity between the membership values of the points of the two sets as well as possible. In Paper III, we investigate defuzzification of a fuzzy disk based on preservation of some geometric inter-relations, when geometric features (in particular, area and radius of a disk) are derived from a fuzzy set.

5.4.1 Geometric features

The defuzzification method presented in Paper VII is based on similarity between a crisp object C and a fuzzy object F. Similarity can be quantitatively expressed in terms of various local and global numerically represented features. Local similarity is measured in each point. It is essential for the localization of the crisp object, since local features are, by nature, position variant. Global similarity is related to (global) geometric properties of the compared fuzzy and crisp object. Inclusion of global properties can be seen as a refinement of the selection procedure. Often being position invariant, global features are used to choose a crisp configuration, by using, among detected well-positioned ones, the configuration providing the highest global similarity. Local features that can be incorporated into the similarity measure are, e.g., membership values, gradients, and curvatures in corresponding points of C and F. Global features that we find interesting to consider are perimeter, area and other moments, centre of gravity, convexity, and/or other shape descriptors.

We demand that similar objects have low point-wise difference in membership values. Intuitively, the points with high memberships to the fuzzy object should be included in its crisp counterpart, and those with low memberships should be assigned to the background. In addition, we use gradient information from a fuzzy segmented image, since high gradient values correspond to the most probable border positions. In order to take the global geometry of the object into account, we consider area, perimeter, and centre of gravity similarity between the objects. In general, which features to chose and how to combine them depends on the particular problem in focus and available knowledge about specific object features.

In Paper VII, we define the similarity measure Ψ as a linear combination of similarity terms

$$\Psi = \sum_{f \in \Phi} \omega_f \Psi_f \ , \tag{8}$$

where the parameters ω_f are weights assigned to the feature similarities Ψ_f from the selected feature set Φ. To simplify comparison, all the similarity terms are defined to have values in the range $[0, 1]$. By adjusting the weights ω_f, the different similarity terms Ψ_f can be taken into account to different extents in the process, in order to make it appropriately adjusted to the application.

The synthetic image presented in Figure 22(a) can be used as an illustration of the effect of each of the observed features on the defuzzification. The shape we observe is composed of an (fuzzy) ellipse inscribed into a (fuzzy) disk, with a radially non-increasing membership function. The image is defuzzified using different combinations of the terms in the similarity function. The kernel of the fuzzy set is indicated with a darker shade of grey in the defuzzified images. The centre of gravity of the fuzzy set and of the defuzzified set are marked with "$\times$" and "$+$", respectively. When the two centres of gravity coincide, the marks overlap and create "$\ast$".

The terms based on membership and gradient, respectively, are position-variant. Hence, they provide good positioning of the crisp object with respect to the fuzzy object, but global geometry is not considered. However, defuzzification based only on the membership term simply provides thresholding of the membership function at 0.5. This is illustrated in Figure 22(b). Defuzzification by using the gradient similarity term only, presented in Figure 22(c), here extracts only the kernel of the object, which corresponds to the position of the strongest edges in this image.

Area and perimeter are position-invariant global features, closely related to the object geometry. It is difficult, if not impossible, to obtain a visually appealing defuzzification when area and/or perimeter are used alone. If area is used as the only parameter to be matched, the crisp object is created by filling the support of the fuzzy object in the order of adding pixel decided by the optimization algorithm, until the desired area is reached, see Figure 22(d). The result if using only the perimeter similarity term is presented in Figure 22(e). In this case, one-pixel-size holes and one-pixel-size components often appear in the defuzzification, since they provide a fast increase in the perimeter towards the goal value. When combined with other features, this undesired tendency of the perimeter term is suppressed.

Centre of gravity (CoG) is a global position-variant feature; it puts constraints both on the position and on the shape of the defuzzification result. This is illustrated in Figure 22(f), where CoG similarity is combined with area similarity; both the positioning and the shape of defuzzification are more similar to the object in Figure 22(a), compared to the result based on area only, Figure 22(d). However, the defuzzification is not at all satisfactory, and is finally improved by combining all the observed features, Figure 22(g). We conclude that the defuzzification result is both well-located and has reliable shape characteristics if both position-invariant and -variant features are used.

In Table 1, the values for different similarity terms in defuzzification of the shape in Figure 22(a) are presented. The defuzzification shown in Figure 22(b)–(g) are considered. This example emphasizes the importance of combining all the features in order to obtain good defuzzification results, mostly due to high fuzziness and a rather wide fuzzy border, present in the shape in Figure 22(a). It is clear that it is

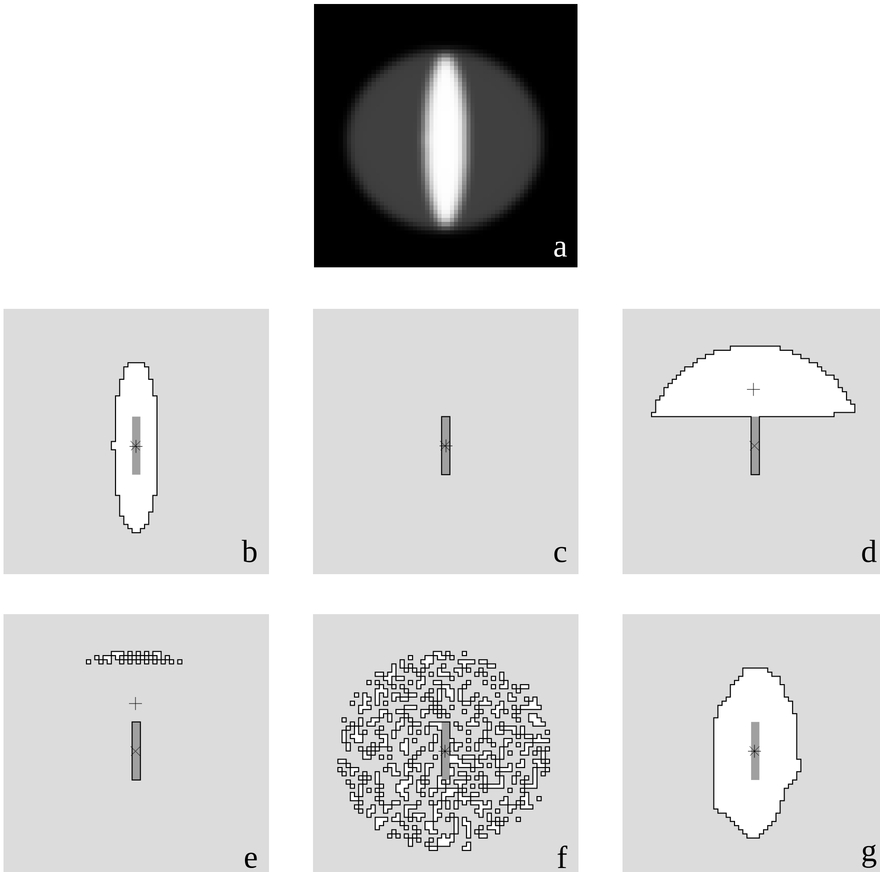


Figure 22: A fuzzy shape (a) and its defuzzification based on different features. Simulated annealing algorithm is applied. (b) Membership similarity. (c) Gradient similarity. (d) Area similarity. (e) Perimeter similarity. (f) Area and centre of gravity similarity. (g) Membership, gradient, area, perimeter, and centre of gravity similarity. In (b)–(g), the grey pixels within the defuzzification are kernel pixels, whereas the white ones are added during the optimization process.

Table 1: Values for the different similarity terms in the similarity function Ψ when defuzzifying the shape in Figure 22(a). Simulated annealing algorithm is used. Features which are used in the optimization (with weights $w_f = 1$) have values written in **bold**. Other features are not considered during the optimization, by setting $w_f = 0$. Ψ_M, Ψ_G, Ψ_A, Ψ_P, and Ψ_{CoG} denote membership, gradient, area, perimeter, and centre of gravity similarity terms, respectively.

Figure	Ψ_M	Ψ_G	Ψ_A	Ψ_P	Ψ_{CoG}	Total
22(b)	**0.806**	0.977	0.707	0.943	0.994	4.427
22(c)	0.686	**0.981**	0.083	0.455	0.992	3.197
22(d)	0.563	0.964	**1.000**	0.848	0.469	3.844
22(e)	0.672	0.968	0.162	**1.000**	0.553	3.355
22(f)	0.576	0.764	**1.000**	0.142	**1.000**	3.482
22(g)	**0.747**	**0.970**	**1.000**	**0.999**	**0.999**	4.715

possible to get high similarity(ies) for only the feature(s) explicitly used in defuzzification, while the other similarities are low, see, e.g., Figure 22(e). Consequently, the defuzzification results may look poor, and not in accordance with what might be expected from the fuzzy image, if insufficient demands (number of features to match) are put on the defuzzification.

In order to find the desired defuzzification, i.e., the crisp object that is the most similar to the fuzzy set, we search for the crisp configuration that maximizes the given similarity function Ψ. Even though the basic idea of the suggested method is straightforward and intuitive, its direct implementation, relying on an exhaustive search procedure, would lead to an extremely high time consumption in any real application. Since this optimization problem is well decoupled from the theoretical idea of defuzzification by feature invariance, almost any optimization method can be adjusted to the task of finding the (global) maximum of the similarity function. We investigated different heuristic search algorithms and presented two of them, floating search (Pudil et al. (1994a,b)) and simulated annealing (Metropolis et al. (1953)). The two methods provide a good trade-off between robustness and speed. Floating search is relatively faster and deterministic, while the non-deterministic simulated annealing can provide higher accuracy at the cost of higher computational complexity, due to cooling scheme that can be designed by a user. By changing the cooling speed of the simulated annealing, the search for the optimal solution can be made more or less thorough. On the other hand, the speed of floating search decreases more significantly with the increase in a search space, than the speed of simulated annealing. The choice of algorithm should be made in accordance with the demands of the task to solve.

Tested on MRA images of human aorta in Paper VII, the defuzzification method shows an ability to deal with the topological properties of fuzzy images. Even though constraints related to the number of components in defuzzification can be incorporated in the procedure (we suggested one method based on preservation of

Table 2: Values for the different similarity terms in the similarity function Ψ when defuzzifying the shape in Figure 23(a).

Figure	Ψ_M	Ψ_G	Ψ_A	Ψ_P	Ψ_{CoG}	Total
23(c)	0.904	0.914	1.000	0.999	1.000	4.817
23(d)	0.908	0.917	1.000	1.000	1.000	4.825

the number of components of the kernel and one method based on manual seeding), it is noticed that the solution selected by the similarity optimization process, when applied without constraints, is often appealing for a human observer. Such an approach provides a good alternative when a priori topology information is unavailable.

The performance of both search algorithms, applied with no additional (topological) constraints to the fuzzy segmented scanning electron microscope image of a cross section of a wood fibre is presented in Figure 23. The image in Figure 23(a) is fuzzy segmented using fuzzy thresholding based on entropy minimization, as described in Pal and Rosenfeld (1988), with a bandwidth $\delta = 50$ (of 255), which preserves a large amount of fuzziness in the images. To remove close lying structures which are not part of the fibre, points of the segmented object are assigned membership value zero, if they have a fuzzy connectedness (Rosenfeld (1984)) to the kernel of the fibre lower than 0.1. The result is presented in Figure 23(b). This image is defuzzified by floating search, Figure 23(c), and simulated annealing, Figure 23(d). The values of the similarity terms for both methods are given in Table 2. We find that both methods provide high similarity with the fuzzy object, and produce visually acceptable result.

5.4.2 Moments

In Paper III, we study the representation and reconstruction of a fuzzy disk by using moments. In connection to that, we analyse the most appropriate defuzzification of an analytically defined fuzzy disk, to a crisp disk. We would like to design defuzzification so that it is an acceptable approximation of an “inverse” of area coverage fuzzification, and to “couple” it with analytical fuzzification which is a good approximation of area coverage. Our study is restricted to disks; to perform defuzzification, it is enough to define the centre position and the radius of the corresponding crisp disk. We choose to follow the idea of preservation of well-known geometric inter-relations for generated defuzzified shapes. The centre of gravity of a (fuzzy) shape is defined in terms of moments. Using the same point to be the centre (of gravity) of both fuzzy and defuzzified shape is equivalent to demanding centre of gravity invariance, as in the method presented in Paper VII. The zero-order moment of a fuzzy shape gives its area. The radius of defuzzification of a fuzzy disk can be defined such that $area = radius^2\pi$ holds, for $area$ obtained from the zero-order moment. Consequently, area invariance is provided, as well as invariance of radii of fuzzy and crisp disks, where the radius is a feature of a special interest in this case.

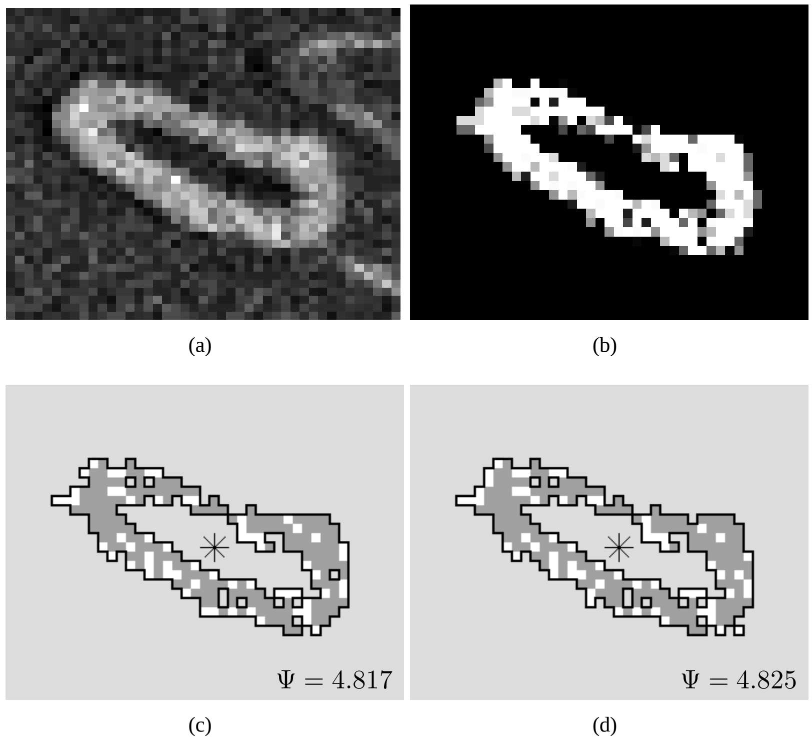


Figure 23: (a) A cross section of a wood fibre in a scanning electron microscope. (b) Fuzzy segmentation of (a). (c) Defuzzification of (b) by floating search. (d) Defuzzification of (b) by simulated annealing. In (c) and (d), the grey pixels within the defuzzification are kernel pixels, whereas the white ones are added during the optimization process.

5.4.3 Remark

The correspondence between fuzzy and crisp spatial sets can be established, and evaluated, in many ways. To perform a thorough evaluation of the defuzzification method that we suggest would require consideration of a variety of fuzzification methods and to perform an analysis of the dependence of the defuzzification results on the input fuzzification. That is an extensive work. In addition, a large number of features and parameters should be tested in many combinations. Instead of carrying out such an evaluation, we have presented some illustrative examples of the performance of the method. An interesting evaluation criterion could be the invariance of the defuzzification to translation of the object within the digitization grid, considering that the estimators of the features chosen to be matched are improved by using a fuzzy shape representation so that they are less sensitive to translation. However, the non-deterministic nature of one of the suggested optimization algorithms (simulated annealing) makes it difficult to derive reliable conclusions regarding this problem. Additionally, a qualitative evaluation of the defuzzification results is of high interest, but being naturally subjective, it is rather difficult to perform it reliably.

One of the approaches to establish the correspondence between fuzzy and crisp spatial sets is to match their selected properties. Our work has taken that direction. We consider it to be consistent with a general approach in the development of fuzzy set theory: properties of fuzzy sets should be defined so that the restriction to the crisp sets fits already existing results from the theory of crisp sets. The remaining “space” in the fuzzy set theory is open for a variety of solutions, both purely theoretically appealing, and coming as demands of practice. We took some steps in that field, finding a combination of theoretical and practical demands to be the most appealing guidance.

6 Brief summary of the included papers

A brief summary, containing the focus and the main results, is given for each of the seven included papers.

Paper I Representation and reconstruction of ellipses from their digitizations are considered. The main result is that the set of digital ellipses is in a one-to-one correspondence with their (finite) moment based representations. This enables both unique coding and recoverability of a digital shape from its code. In addition, error bounds for the approximate reconstruction of a continuous ellipse from the moments of its digitization are derived. The results are extended to 3D.

Paper II Error bounds for the estimation of moments from a fuzzy representation of a shape are derived and compared with the estimations from a crisp representation. It is shown that a fuzzy membership function based on pixel area coverage provides higher accuracy, compared to binary Gauss digitization at the same spatial resolution. The errors of the estimates decrease both with increased size of a shape (spatial resolution) and increased membership resolution (number of available grey-levels).

Paper III Representation and reconstruction of fuzzy disks by using moments are analysed, in both the continuous and the discrete case. It is shown that for a certain class of membership functions defining a fuzzy disk, there exists a one-to-one correspondence between the set of fuzzy disks and the set of their generalized moment representations. Theoretical error bounds for the accuracy of the estimation of moments of a continuous fuzzy disk from the moments of its digitization and, in connection with that, the accuracy of an approximate reconstruction of a continuous fuzzy disk from the moments of its digitization, are derived.

Paper IV Shape description by the shape signature based on the distance of the boundary points from the shape centroid is analysed and extended to the case of fuzzy sets. An analysis of the transition from a crisp to a fuzzy shape descriptor is given for both the continuous and the discrete case. The specific issues induced by using a discrete representation of the objects are of the highest interest. Two methods for calculating the signature of a fuzzy shape are considered. The methods are derived from two ways of defining a fuzzy set: first, by its membership function, and second, as a stack of α-cuts. A statistical study, characterizing the performance of each method in the discrete case, is done.

Paper V An initial investigation of the perimeter, area, and P^2/A estimations, derived from a fuzzy representation of a discrete shape, is performed. A novel method is developed for the estimation of perimeter of a discrete fuzzy shape. The performance of the suggested estimators is studied for digitized disks and digitized squares, fuzzified by the area coverage approach. The suggested estimation methods exhibit much better performance than the results obtained from the crisp shape representation, especially in the case of low resolution images.

Paper VI The results presented in Paper V are further improved and the precision of the estimation of the perimeter, area, and P^2/A measure, obtained from a fuzzy representation of a discrete shape is increased. The study is extended to 3D and the performance of surface area, volume, and roundness measure estimators for digitized balls with fuzzy borders is analysed. It is shown that the suggested method provides significant improvement in precision, compared to analogous estimation results obtained from a crisp shape representation, especially at low resolutions.

Paper VII A defuzzification method based on the invariance of feature values between fuzzy and crisp representations is presented. The method produces crisp shapes from fuzzy shapes, while keeping selected properties of the two representations as similar as possible. Combining two defuzzification approaches, we utilize the information contained in the fuzzy representation both for defining a mapping from the set of fuzzy sets to the set of crisp sets, and for an approximate reconstruction of an unknown crisp original. We suggest two search algorithms for optimizing similarity based on perimeter, area, membership values, gradient, and centre of gravity of the two shapes.

7 Applications

The work presented in Papers I–VII is performed independent of any particular application. Nevertheless, the possible applications are numerous. We have been working on improving accuracy and precision of some well-known shape descriptors and our results could be applied wherever precise estimates of shape properties are required, especially in conditions of limited spatial resolution. We showed, either theoretically, or empirically, that fuzzy segmented images provide valuable information contained in grey-levels, and that so-called membership resolution available can be successfully utilized to overcome the lack of spatial resolution.

While testing our methods, we used synthetic shapes, with known values of the parameters to be estimated (perimeter, area, signature, moments). Tests on real images usually allow only subjective evaluation, since the ground truth is seldom available. Having a ground truth data is essential when comparing the performance of different methods and when evaluating new ones. The performance of our perimeter estimation, as well as of the signature calculation, are evaluated by the statistical studies of fuzzy disks. Our moment estimation error analysis is performed and proved theoretically, and additionally supported by empirical results obtained for fuzzy disks and fuzzy squares.

After having shown the improved performance of the presented methods based on fuzzy sets, we suggested some possible applications. In Paper III, we observe fuzzy disks defined by analytical functions, and show their similarity with fuzzy disks obtained by (theoretical) area coverage fuzzification. Consequently, our theoretical study is put in connection with fuzzy shapes obtained from real imaging situations, since many imaging devices produce images with grey-levels of pixels based on area coverage. As an example, Figure 24 shows fuzzy digital objects obtained by different approaches: (a) fuzzification by a piecewise linear non-increasing function; (b) fuzzification by discrete area coverage approach; (c) imaging of a circular object by a scanner set; (d) MRA imaging – a cross-section of a human aorta. We find that the synthetic ones, (a) and (b), are visually similar to (c) and (d), resulting from real imaging processes. Consequently, we hope that the methods developed and tested on synthetic images like, e.g., (a) and (b), will find applications in the analysis of real images like, e.g., (c) and (d).

Paper VII presents one application of the improved perimeter and area estimation methods, described in Papers V and VI. Defuzzification based on feature invariance relies on improved estimations of measures of the observed shape. We applied the defuzzification method to several real images, and found the defuzzification results good, both visually and in terms of numerical evaluation. One of the examples is defuzzification of a fuzzy segmented scanning electron microscope image of a cross-section of a wood fibre, shown in Figure 23. Another example is defuzzification of a part of a histological image of a bone implant (inserted in a leg of a rabbit), presented in Figure 25. The original is the colour image acquired by a light microscope. We tested our method on a part of the image, containing a bone area (in purple/bluish), surrounded by a non-bone area (light).

Our opinion is that our defuzzification method, following fuzzy segmentation,

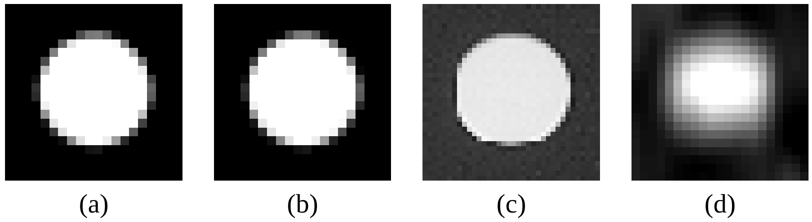


Figure 24: Fuzzy discrete objects. (a) Fuzzy discrete disk obtained by using a piecewise linear non-increasing function. (b) Pixel area coverage based fuzzification of a disk. (c) Image of a circular hole in dark paper, obtained by a scanner set. (d) Cross-section of an MRA image of a human aorta.

can be successfully used in rather difficult crisp segmentation tasks where topological properties of fuzzy sets are not easily interpreted in terms of crisp sets. As an example, the defuzzification of three consecutive slices of a three-dimensional MRA image of a human aorta is shown in Figure 26(a). The three slices display the region where the aorta separates into the two iliac arteries, leading from the abdomen into the legs. The decision on how many components there exist in each slice is clearly subjective and highly dependent on the contrast settings. In this specific case, knowledge about the image content and human anatomy does not provide any additional help. We find that the solution selected by the similarity optimization process, when applied without constraints, is appealing for a human observer and can be seen as a good defuzzification (crisp segmentation) result. Such an approach provides a good alternative when a priori topology information is unavailable. Alternatively, existing a priori knowledge about topology of the object can be incorporated into the process.

Our hope is that all the methods presented in Papers I–VII will show applicability in practice. We would like to continue our work not only in further theoretical development of the analysis of fuzzy spatial sets, but also in finding ways to use the theoretical results in solving real image analysis problems.

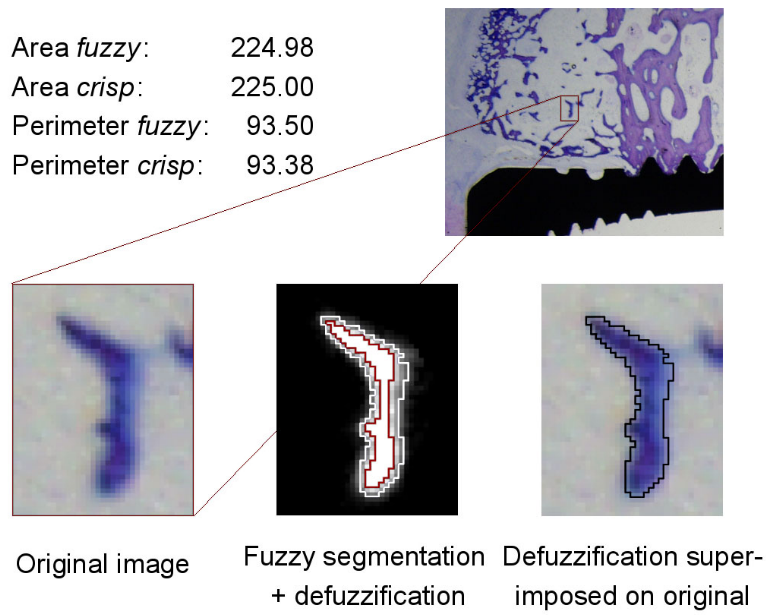
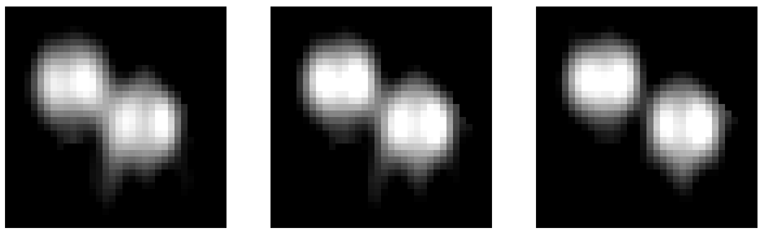
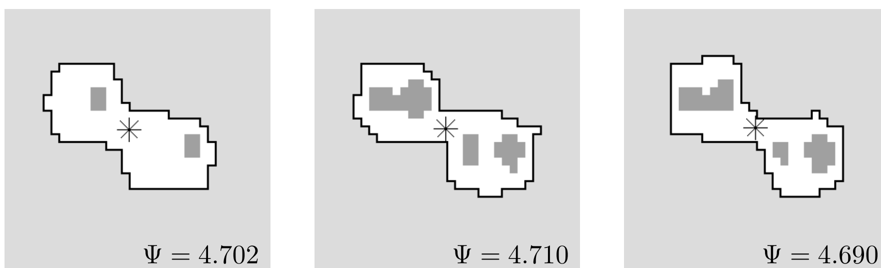


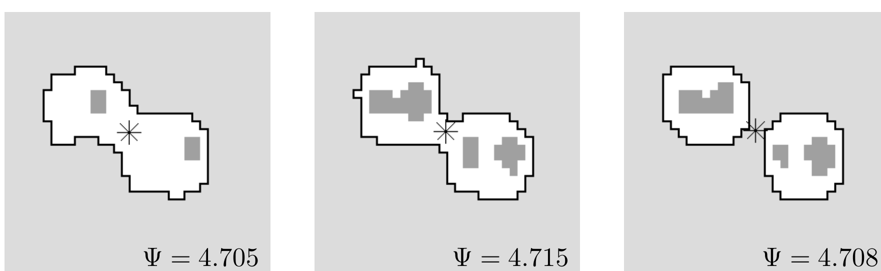
Figure 25: Defuzzification of a selected part of the microscope image of a bone implant. The implant is the black shape in the colour (grey-level) image. The purple/bluish (dark grey) area is bone, the light parts (light grey) are non-bone areas.



(a) Fuzzy segmented vessels from an MRA image



(b) Defuzzification by floating search



(c) Defuzzification by simulated annealing

Figure 26: Defuzzifications of three vessel slices using no topological constraints. The similarity Ψ, defined as in Equation (8), is given for each solution. Weights are set to 1 for all the terms. Kernels of the fuzzy sets in (a) are shown in grey in (b) and (c).

8 Closing remarks

Our main hypothesis, that *fuzzy spatial sets can provide improved descriptions of discretized continuous shapes*, compared to representations based on classically used crisp sets, has been confirmed by the work presented in this Thesis. In addition, it has become clear that there are many specificities of a discrete space which often lead to difficulties when results from the continuous space are to be adjusted to the discrete case. We have suggested some approaches appropriate for the analysis of discrete fuzzy spatial sets. However, every question we answered lead to new questions that we would like to answer.

8.1 Future work

Our results regarding moments of a fuzzy shape, presented in Papers I–III, are obtained for restricted classes of shapes. The results related to moment estimations hold for the shapes whose boundaries are piece-wise 3-smooth curves, but the reconstruction of the shapes is analysed only in cases where exists relatively simple functional dependence between relevant parameters of the shape and its moments. We are interested in extending our study to more general shapes, but also in using other types of moments, than geometric ones. The moments defined on orthogonal bases (e.g., Zernike moments) are of special interest. Orthogonal moments have more simply defined inverse transforms, which provides easier recovering of a shape than from the geometric moments. It would be interesting to investigate their behaviour in representation and reconstruction of fuzzy shapes.

Further exploration of the properties of theoretically defined fuzzy sets and their connections with shapes appearing in imaging is also a challenge. A variety of fuzzy membership functions, that can be used to mathematically define fuzzy shapes, enables approximation of different real shapes, coming from different imaging conditions and applications , with some appropriately chosen mathematically defined fuzzy shape. The knowledge about mathematical shapes, easy to derive and work with, can be used in the analysis of real, more complex, shapes, present in real world applications. We find it worth an effort to put in connection fuzzy shapes resulting from fuzzification functions and fuzzy shapes resulting from imaging by particular imaging devices.

The results related to shape signatures, given in Paper IV, are obtained for the class of star-shaped fuzzy sets. We think that the power of a fuzzy approach will be even better exhibited when extended to more general cases and used to deal with topology-related issues (like, e.g., “internal” and “external”). In connection with that, it might be of interest to explore various fuzzy distance functions, to be possibly incorporated into the definition of a signature function.

More generally, it is of interest to explore deeper how the definitions of various concepts of interest, when adjusted or introduced to be applied to discrete fuzzy sets, behave depending on which of the two representations of fuzzy sets – (1) by its membership function, and (2) as a stack of its α-cuts – is used. Two examples, the signature and the perimeter of a fuzzy shape, exhibit different behaviours. The signature, calculated as suggested in Paper IV, commutes with the integration within the fuzzification (the integrated signatures of all α-cuts give the same result as the signature calculated by integrated distances) in the continuous case, while the same does not hold in the discrete case. On the other hand, the perimeter of a fuzzy shape can be calculated either by integrating perimeters over the α-cuts of a shape, or by using a membership function directly; results are the same, which holds both for the continuous and the discrete case. A more general conclusion with respect to this question is desired.

Geometric shape descriptors, studied in Papers V and VI, could be revisited, while putting into focus the approach suggested by Bogomolny (1987). More generally, the study of fuzzy sets can be done with stronger emphasis on their inner properties. The main interest of our studies, so far, has been to utilize information present in a fuzzy representation of a shape in order to obtain improved estimations of various shape measures. We feel that there are more possibilities to exploit the information-rich fuzzy representations.

A theoretical justification of the statistically confirmed results on perimeter and surface area estimations should be derived.

In our theoretical studies, one of the prior future tasks is to utilize fuzzy numbers as measures of fuzzy sets. An appealing direction to take is also towards studies on topological properties of fuzzy discrete sets.

Defuzzification methods can be further developed. The similarity function can be modified to incorporate some additional/alternative features. With respect to that, the shape signature and the moments of higher order are of interest, but also any other feature which can be estimated with high precision from the fuzzy representation of a shape. The questions of an optimal number of features to use, or their appropriate choice, could be interesting to address, as well. A modification of the similarity function so that the features are incorporated in it in some other way than as a linear combination is considered. Initial steps in that direction have already been taken.

Another direction to take is to perform defuzzification based on feature similarity in such a way that the crisp object is generated at a higher spatial resolution than the fuzzy object used as a source of the estimates for the selected features (Borga (2005)). Our results presented in Papers II and III show that the precision of the moments estimations from a fuzzy shape representation is higher than the precision obtained if a crisp representation is used at the same spatial resolution. If similar results can be proven for other features of interest, defuzzification at an increased spatial resolution, with the preserved precision of the feature estimates, is a natural consequence and a reasonable goal. It is interesting to investigate what features can easily be adjusted and used in the optimization of the similarity between two shapes, when the goal is to reconstruct the shape at sub-pixel resolution. A theoretical study of the optimal increase in the resolution in such a reconstruction is desired.

Application of the developed methods to image analysis tasks coming from the real world is, certainly, of a great interest. The pleasure we have had while working with fuzzy sets will be even bigger if the methods we considered find their use in practice. We have no doubts that it will happen.

8.2 Closing words

The wonderful richness of the world around us is in nuances and variations of everything that appears in it. Hardly anything is just “black” or “white”. It is possible to avoid imposing hard divisions and extreme decisions, so often undesired and misleading, through exploring of soft transitions and grading between the two extremes, and through incorporating the levels of tolerance and the huge potential of “in-between” solutions.

To learn to understand the infinite power of different compositions of colours, shapes, materials, impressions, feelings, views, and goals, and to take the best from them, is a challenge we find worth putting an effort in, both in our work within image analysis, and in all other aspects of our life.

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The following conferences have been attended, first five before becoming a PhD student:

- VIII International Conference on Logic and Computer Science, Novi Sad, Yugoslavia, September 1997. (Oral presentation)
- 7th International Conference on Discrete Geometry for Computer Imagery (DGCI), Montpellier, France, December 1997. (Oral presentation)
- XIII Conference on Applied Mathematics, Igalo, Yugoslavia, May 1998. (Oral presentation)
- XIV Conference on Applied Mathematics, Palić, Yugoslavia, May 2000. (Oral presentation)
- 9th International Conference on Discrete Geometry for Computer Imagery (DGCI), Uppsala, Sweden, December 2000. (Poster presentation)
- Swedish Society for Automated Image Analysis Symposium on Image Analysis, Stockholm, Sweden, March 2003. (Oral presentation)
- 11th International Conference on Discrete Geometry for Computer Imagery (DGCI), Naples, Italy, November 2003. (Poster presentation)
- 17th International Conference on Pattern Recognition (ICPR), Cambridge, United Kingdom, August 2004. (Poster presentation)
- 13th International Conference on Image Analysis and Processing (ICIAP), Cagliari, Italy, September 2005. (Poster presentation)

Acknowledgements

For a number of years of my postgraduate studies, I was trying, at the same time, to be a teaching assistant at the Faculty of Engineering, and above all, to be a mother. I would have never succeed if I were not fortunate enough to be a daughter, a wife, a sister, a colleague, and a friend, too.

There are many people whose support and efforts are in-built in my work. I would like to express my deepest gratefulness to those without whom this thesis would not be at all.

✿ Gunilla Borgefors, my supervisor, for having trust and giving me a chance. For finding ways to support my studies at CBA in many situations when that required personal efforts. For understanding and accepting my schedules. For giving a valuable scientific support. For offering her huge knowledge and experience.

✿ Ingela Nyström, my assistant supervisor, for generously offering such a nice research idea, as the fuzzy shape analysis is, to me. For introducing her coworkers to me, and making them my co-workers too. For pushing me hard (it payed off ☺). For proof-reading. For accepting my ways. For making space for me, in her home and in her life. For taking me to the North, and coming to me to the South. For sharing her family with me.

✿ Stina Svensson, for being like an guardian angel, since Montpellier 1997. For introducing me to Gunilla and Ingela. For making her home mine. For making my birthday cake. For being patient with me in times hard for her. For proofreading. For lots of good advice, regarding work and life. For all the talking, and understanding. For precious friendship.

✿ Joviša Žunić, my master's thesis supervisor, for all the energy, knowledge, support, and care, invested in me and my work. For introducing the world of research to me, for teaching me digital geometry, and how to read, write, and submit papers. For sending me out in the world.

✿ Joakim Lindblad, for teaching me so many things, with infinite patience and calmness. For fighting my fights with computers. For working with me and making work such a fun. For having trust in me and encouraging me to have it, myself. For infinite amount of proof-reading, and all my corrected mistakes. For endless support. For listening and understanding. For letting me close.

✿ Jocelyn Chanussot, my co-author and my friend. For his tolerance, for his spirit, for his charm. For wine and cheese. For all the fun we have.

✿ Punam K. Saha, my co-author, for his trust in my work and his scientific support.

✿ Lena Wadelius, for having the answers or knowing where to look for them. For making everyday life of a foreigner so much easier.

✿ Olle Eriksson, for not at all giving me any reason for complains on my computer. All faults, unfortunately, could have been only mine.

✿ Ida-Maria Sintorn, for her shining smile and contagious optimism. For always seeing options, and reminding me that the choices are mine. For sharing home with me. For introducing Mathias to me. For all the help, in small and big things. For never letting me feel lonely, but always knowing when I want to be alone.

✿ Felix Wehrmann for warmest friendship. For taking care. For skating.

✿ Xavier and Liliana (Swedish course was such a fun with you!), Carolina and Anders (your invitations for dinners at your place made me feel so happy and "accepted in the society"!), Erik (Finish disco was such an artistic success of ours ☺), Patrick (it was so good that you took me to Stallet to exercise a bit, and to have your Volvo waiting at Arlanda ☺), Lucia (it was very nice to chat with a Southern European ☺ who in addition makes so good cakes!), Bosse (how dull Rullan lunches would have been, without you!), Amalka (it was such a pleasure to be with you in Cagliari!), for making Uppsala a warm place where I always come with a smile, and leave with a hope that I will soon be back again.

✿ Petra, Tommy, Ewert, Ola, Robin, Mats, Hamed, Kristin, Pasha, Magnus, Maria, Filip, Andreas, Suthakar, Anna, Mattias, for their valuable contribution to the excellent atmosphere at CBA.

✿ Mila Stojaković, head of Department for mathematics at Faculty of Engineering, for making my studies in Sweden possible by accepting my leaves. For all the effort put in planning teaching schedules, to make them doable for me.

✿ Silvia Ghilezan and Vanja Pantović, my colleagues and friends, for invaluable moral and scientific support.

✿ Zorica Uzelac and Nebojša Ralević, for not hesitating to overtake big parts of my responsibility for our courses, in times of my absence.

✿ Silvia Likavec, and Lidija Čomić, for proof-reading. For sharing belief that it is worth an effort to go out and try.

✿ Biljana Mihajlović, Tibor Lukić, Ksenija Doroslovački, Ljiljana Teofanov, Vlada Ćurić, Aca Nikolić and Jelena Ivetić, my colleagues who have shared classes with me at various courses, during times of my absences, for always finding strength to follow my over-demanding schedules of comings and goings. For keeping it all going on, and above all - for never showing any sign of being tired of all the extra work.

✿ Zoran, for functional computers. For on-line assistance, whenever needed.

✿ All my colleagues from the Department of mathematics, for endless exams covered without me.

- Ivana and Vesna, English teachers, for solving many of my language dilemmas during these years. For all those application forms, requests, and formal letters that would be written so much worse if it was not for them.
- Lela, our secretary, for all those small everyday things she does all the time to make our work smoother.
- Dragan, for his love, and all the support, in spite of times being hard for him, and in spite of times being hard for both of us. For my peace while I am away, since I know that our girls are safe and happy with him.
- Nebojša Rakočević, for all those hours spent on my computer/programming problems. For making me feel I have a brother.
- Ivana, my sister, for making me confident that there will always be someone who will stand for me. Anywhere. Anyhow.
- Saveta and Ilija, my parents, for invaluable support, during all my life. For accepting in advance to provide any help that might be needed, and never hesitating to make any effort required to keep the promise. For always being around, for making me feel safe.
- Dunja and Marina, my daughters, for existing, and being the way they are. For accepting on my word, with no questions, at the age of 6 and 4, that all the months spent without me are for a good reason. For reminding me that I have to work (if I want to finish ☺), at times of loosing motivation. And for reminding me that I need rest sometimes, too. For understanding how important and precious for me is to hear their laughter over the phone, in times of being away. For making it worth any effort to accomplish these studies. For giving endless inspiration for all the time that comes.

Hvala vam, od sveg srca.

Наташа

Nataša Sladoje

Novi Sad and Uppsala, October 2005