Balanced nested row column designs with binary rows and columns

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ABSTRACT. Some balanced nested row-column designs with p rows, q columns and v treatments such that pq = av + d for some integers a≥1 and d<v are given. They are either universally optimal (BNRC where treatments are replicated p times or not at all in a block) or what is called balanced extended block row-column designs (BEBRC) where treatments are replicated a or a+1 times within block. Many of these designs are cyclic. Average pairwise variances (APVs) are compared for the two types of designs and a formula given for when one of them has smaller APV. It has been common in the literature to consider a = 0 only i.e. to compare BNRC and BIBRC.

1. Introduction

In nested row column designs, NRCDs, experimental material is divided into b blocks and the blocks in a number of rows and columns. We take it to be p rows and q columns in each block. On these bpq experimental units v treatments are compared. The point is that we can eliminate possible variation between blocks, rows and columns, i e compare treatments within rows and columns within blocks.

A simple example is when comparing tire brands. Cars are then blocks with front and rear wheels corresponding to rows, left and right wheels to columns. If we are interested in v = 3 only we could use a design with b=3, p=q=2:

```
1 2    2 3    3 1
2 1    3 2    1 3
```

or a multiple of it. As we will see this is a balanced design and in fact in a certain sense the best possible. It is said to be universally optimal and denoted by BNRC.

In a NRCD treatment i is replicated r_{ijkl} times in block j, row jk and column jl.

We will consider the case with binary rows and columns, that is r_{ijkl} is either 1 or 0.

The block replications $r_{ij} = \Sigma_{kl} r_{ijkl}$ in the example above are either 2 or 0. This is not optimal if variations between rows and columns are found to be so small that we think of ignoring them in the analysis. Therefore it is of interest to compare BNRC to designs where the r_{ij} differ by at most 1 within block. Such a design for the situation above would be

```
1 2    2 3    3 1
3 1    1 2    2 3
```

This is a balanced nested row column design with extended blocks and is denoted BEBRC.

The two designs above are cyclic and can be specified by their first (initial) block.

The purpose of this note is to find and compare BNRC and BEBRC for some values of v, b, p and q.

2. Balanced and universally optimal designs.

The usual model for the observation vector **y** is written:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}_b\boldsymbol{\beta} + \mathbf{Z}_{br}\boldsymbol{\gamma} + \mathbf{Z}_{bc}\boldsymbol{\delta} + \mathbf{e}$$

where **X** is the bpq*v incidence matrix for treatments and Z_b, Z_{br}, Z_{bc} the corresponding matrices for blocks, rows within blocks and columns within blocks. They have b, bp and bq columns respectively. The residual vector **e** has mean **0** and variance covariance matrix $\sigma^2\mathbf{I}$ where **I** is the identity matrix.
In the fixed case where we estimate treatment effects $\boldsymbol{\alpha}$ within rows and columns within blocks only we get the estimation (reduced normal) equations $\mathbf{A}_4\boldsymbol{\alpha} = \mathbf{Q}$ where

$$\mathbf{A}_4 = \mathbf{X}`\mathbf{X} - (1/q)\mathbf{N}_{br}\mathbf{N}`_{br} - (1/p)\mathbf{N}_{bc}\mathbf{N}`_{bc} + (1/pq)\mathbf{N}_b\mathbf{N}`_b$$

Here $\mathbf{N}_b = \mathbf{X}`\mathbf{Z}_b$ etc. Now the design is said to be balanced if $\mathbf{A}_4$ is completely symmetric, i e on the form c**I** – d**J** for some constants c and d and where **J** is a matrix of '1's. Moreover if tr($\mathbf{A}_4$) is maximized the design is said to be universally optimal.
For the illustration in the previous paragraph we have $\mathbf{A}_4$ = .75**I** - .25**J** (BNRC) and =.5625**I** - .1875**J** (BEBRC).
Bagchi et al. (1990) has proved that a BNRC is obtained if

1. columns form a balanced (here) incomplete block design
2. (as rows are binary) the same treatments are repeated in all the rows of the same block.

The second requirement means that $(1/q)\mathbf{N}_{br}\mathbf{N}`_{br} = (1/pq)\mathbf{N}_b\mathbf{N}`_b$. Thus blocks and rows need not be balanced. If they are the design is said to be of series A (Morgan&Uddin, 1993).
Designs are now searched for pq = av + d, $1 \le a \le p$, $0 \le d < v$, $p \le q < v$ where a and d are some integers. When a = 0 BNRC is compared to BIBRC . This is thoroughly studied by Morgan & Uddin (op cit.). Here the cases a = 1, d > 0 and a > 1 will be discussed.
We will now also use $\mathbf{A}^*_4 = \mathbf{A}_4/r$ where $r=\Sigma_j r_{ij}$ (John&Williams 1995). It can be written:

$$v(p-1)/(p(v-1))\mathbf{I} - (p-1)/(p(v-1))\mathbf{J} \quad \text{for BNRC and for BEBRC}$$

$$(vpq(pq-p-q+1+2a) - a(a+1)v^2)/(p^2q^2(v-1))\mathbf{I} - \\ - \quad (pq(pq-p-q+1+2a)-a(a+1)v)/(p^2q^2(v-1))\mathbf{J}$$

The v-1 positive eigenvalues of A* are called canonical efficiency factors and their sum equals trace(A*). In the balanced case they are all equal and thus for

BNRC: v(p-1)/(p(v-1) = bpq(p-1)/(rp(v-1) and for

BEBRC: $(vpq((p-1)(q-1)+2a)-a(a+1)v^2)/(p^2q^2(v-1)) =$
$b[(p-1)(q-1) + \{2apq-a(a+1)v\}/pq]/(r(v-1))$

The relative efficiency of BEBRC to BNRC is therefore

$$[(p-1)(q-1)+\{2apq-a(a+1)v\}/pq]/(q(p-1))$$

if fixed blocks, rows and columns are kept in the model.

With a = 0 this is = (q-1)/q as given in Morgan & Uddin for BIBRC and it is increasing with a.

3. Recovery of information from all strata

When block, row and column effects are considered random we write the model equation as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}_b\mathbf{b} + \mathbf{Z}_{br}\mathbf{g} + \mathbf{Z}_{bc}\mathbf{d} + \mathbf{e}$$

where the independent random vectors **b**, **g**, **d**, **e** have zero means and variance covariance matrices $\sigma^2_b\mathbf{I}_b$, $\sigma^2_{br}\mathbf{I}_{bp}$, $\sigma^2_{bc}\mathbf{I}_{bq}$, $\sigma^2\mathbf{I}_{bpq}$.

The anova-table is then

Source	Df	Sum of squares	Expected mean squares
Blocks	b-1	$y`\mathbf{A}_1y$	$w_1 = \sigma^2 + p\sigma^2_{bc} + q\sigma^2_{br} + pq\sigma^2_b$
(Blocks)Rows	b(p-1)	$y`\mathbf{A}_2y$	$w_2 = \sigma^2 + q\sigma^2_{br}$
(Blocks)Cols	b(q-1)	$y`\mathbf{A}_3y$	$w_3 = \sigma^2 + p\sigma^2_{bc}$
Residuals	b(p-1)(q-1)	$y`\mathbf{A}_4y$	$w_4 = \sigma^2$

When the variance components are known we can use the method of generalized least squares to estimate $\boldsymbol{\alpha}$ which means that we recover information from strata 1-4 above.
The variance covariance matrix for this estimator is the inverse of $\mathbf{VI} = \Sigma_i\mathbf{A}_i/w_i$ (John&Williams 1995). In a series-A design all $\mathbf{A}_i = c_i\mathbf{I} + d_i\mathbf{J}$ and thus we can write $\mathbf{VI} = T\mathbf{I} + U\mathbf{J}$ and $\mathbf{V} = \mathbf{VI}^{-1} = T^{-1}\mathbf{I} + M\mathbf{J}$ for some M. Thus the pairwise variances are all = 2/T where T = $\Sigma_i c_i/w_i$.
If we can find the c_i-values for BNRCD and BEBRCD we can compare the efficiencies of the two types of designs.
It is quite easy to find the main diagonal terms of $\mathbf{A}^*_i$ as follows:

	BNRC	BEBRC
$\mathbf{A}^*_1$	(v-q)/vq	$\{pq(1+2a)v-a(a+1)v^2 – p^2q^2\}/(vp^2q^2)$
$\mathbf{A}^*_2$	0	$\{pq(p-1-2a) + a(a+1)v\}/(p^2q^2)$
$\mathbf{A}^*_3$	(q-p)/qp	$\{pq(q-1-2a) + a(a+1)v\}/(p^2q^2)$
$\mathbf{A}^*_4$	(p-1)/p	$\{pq((p-1)(q-1) + 2a) – a(a+1)v\}/(p^2q^2)$

The off-diagonal terms = - (main-diagonal terms)/(v-1) and c_i = (main-diagonal term) – (off-diagonal term) i.e. = v(main-diagonal term)/(v-1). Now we can compute the T-values (T_{BN} and T_{BE}) for the two types of design and we have that BNRC is superior to (has a smaller variance then) BEBRC if $T_{BN}/T_{BE} > 1$.

After some calculations we can see that this is the same as

$$C(w_2w_3w_4 - w_1w_2w_4 + w_1w_2w_3 - w_1w_3w_4) > 0$$

where $C = pqv(p-1-2a) + a(a+1)v^2$, a positive number that can be dropped. Except for C this is the same as that obtained by Morgan & Uddin (1993) when comparing BNRC and BIBRC . The criterion for BNRC-superiority is then seen to be

$$w_1 > \sigma^2[\{(\sigma^2+q\sigma^2_{br})(q\sigma^2_{br}+pq\sigma^2_b)\}/\{q\sigma^2_{br}(\sigma^2+p\sigma^2_{bc})\}$$

which can be further simplified to

$$w_1 > \sigma^2[(\sigma^2\sigma^2_b)/(\sigma^2_{br}\sigma^2_{bc}) - 1]$$

given by Morgan & Uddin (1993).
As an illustration we can take p=2, q=4, $\sigma^2=1,\sigma^2_b=2,\sigma^2_{bc}=.4$. Changing σ^2_{br} it is seen that for .25240274 w_1 is equal to the right hand side. Increasing (decreasing) σ^2_{br} makes BNRC superior (inferior). The same is true when decreasing (increasing) σ^2_b.

4. Designs for 3 $\le$ v $\le$ 7 and pq > v.

BNRC
For a NRCDvbpq with binary rows and columns to be BNRC it is necessary that the bq columns of size p constitute a BIBD. Therefore bq/v and bpq(p-1)/(v(v-1)) must be integers and of course such a BIBD must exist. Moreover it must be possible to divide this BIBD into b parts with just q treatments within each part.
For the design to belong to series A rows must constitute a BIBD why bq(q-1)/(v(v-1)) must be an integer.
As an example take (v,p,q) = (7,3,5). Then 5b/7 (and 30b/42 = 5b/7) must be an integer, i. e. b must be a multiple of 7. For series A b must be at least 21 as 20b/42 must be an integer.
A BNRC7.7.3.5 do exist, cyclic with initial block:

1	2	4	5	7
2	7	5	4	1
4	5	1	7	2

Columns are balanced as their within-column-differences are 1,3,2,2,3,2,1,3,3,1,2,3,1,2,1.
Rows have differences 1,3,3,1,2,3,2,1,3,2 and are not balanced. The design does not belong to series A.

But if we include 14 more cyclic blocks with the two initial blocks:

1 2 3 4 6	1 2 3 4 5
3 6 4 2 1	5 4 2 1 3
2 3 6 1 4	4 3 5 2 1

we obtain a series A BNRC7.21.3.5.

BEBRC

Series A means that blocks, rows and columns are balanced for which it is necessary that bd/v, bd(d-1)/(v(v-1)), bpq/v, bpq(q-1)/(v(v-1)) and bpq(p-1)/(v(v-1)) are integers.
With (v,p,q) = (7,3,5) we have d = 1 and see that b = 7 could be enough. In fact a cyclic BEBRC7.7.3.5 is obtained from the initial block:

```
1 2 3 4 7
5 1 2 6 4
6 7 5 1 3
```

An example of a BEBRC that does not belong to series A is the following with (v,b,p,q) = (6,10,3,3):

```
6 1 4    5 3 4    5 1 4    1 3 5    5 1 6
1 4 2    1 6 2    4 3 2    5 1 3    6 2 1
3 6 5    6 2 5    6 2 3    6 4 2    3 4 2

3 1 6    2 5 6    5 4 2    5 4 3    3 4 6
1 2 3    4 1 3    3 2 1    6 1 4    5 1 4
2 5 4    6 3 5    4 6 5    2 5 1    2 6 3
```

Rows and columns within block are not balanced.
A number of designs are found and their parameters presented in Table 1.

Construction

Designs are obtained by trial and error. The minimum number b of blocks is given above. For BEBRC it is as a rule determined by the requirement for block balance and then often by the unreduced BIBD.
With help of a computer it is quite easy to see if there exists a cyclic design. If not, the block, row and column BIBDs are combined with some effort. BEBRC6.15.4.5 is obtained in this way. With d = 2 the two extra treatments per block can be taken as the unreduced BIBD. This is obtained as a cyclic design with initial blocks 1 2, 1 3 and 1 4. The first two parts are full sets with six blocks each, the third a half set. Including all treatments we can use the two initial blocks:

```
1 2 3 4 5    1 2 3 4 5
3 4 2 6 1    4 6 2 3 1
6 3 1 5 2    3 1 4 5 6
2 6 5 1 4    6 4 5 1 2
```

in the first and third cases which both have rows and columns balanced.
But in the second case no such initial block can be found. The remaining six blocks are then constructed so that the six possible rows (in the unreduced BIBD) occur 10 times each among the total of 4*15 = 60 rows and the 15 possible columns five times each among the 75 columns. As first and third cases are balanced there are four times for each row and two times for each column left for the last six blocks, a task which does not require too many trials.

Table 1: Minimal number of blocks for found BNRC and BEBRC. Cyclic designs are indicated by a ‘c’.

v,p,q	a,d	BNRC	BEBRC	v,p,q	a,d	BNRC	BEBRC
3,2,2	1,1	3 c	3 c	7,2,4	1,1	21 c	21 c
4,2,3	1,2	4 c	6 c	7,2,5	1,3	21 c	21 c
4,3,3	2,1	4 c	4 c	7,2,6	1,5	7 c	21 c
5,2,3	1,1	10 c	10 c	7,3,3	1,2	7 c	21 c
5,2,4	1,3	5 c	10 c	7,3,4	1,5	7 c	21 c
5,3,3	1,4	10 c	10 c	7,3,5	2,1	7 c*	7 c
5,3,4	2,2	5 c	10 c	7,3,6	2,4	7 c	7 c
5,4,4	3,1	5 c	5 c	7,4,4	2,2	7 c	21 c
6,2,4	1,2	15	15	7,4,5	2,6	21 c	21 c
6,2,5	1,4	6	15	7,4,6	3,3	7 c	7 c
6,3,3	1,3	10	10*	7,5,6	4,2	7 c	21 c
6,3,4	2,0	15 c	5	7,6,6	5,1	7 c	7 c
6,3,5	2,3	6	10				
6,4,4	2,4	15 c	15				
6,4,5	3,2	6 c	15				
6,5,5	4,1	6 c	6 c				

* Not series A.

Initial blocks for cyclic designs and non-cyclic designs written out in full are given in the Appendix.

5. Alternative cyclic designs with v blocks

In a universally optimal design (BNRC) the matrix A_4 is completely symmetric with maximum trace. This may require quite many blocks and is not always a cyclic design. Moreover, as has been discussed by Bailey & Williams (2007), it is important to consider the qualities of the row as well as the column component designs. That is if the variation between rows and/or columns within blocks may be ignored in the analysis. Therefore BEBRC is a competitor. It may even be that an unbalanced design with better row or column component designs is preferable.
Therefore it is interesting that, for binary rows and columns, there is always a cyclic design v,v,p,q with maximal trace(A_4), see Chang & Notz (1994). Such a design is easily obtained; we can take any q of the v treatments in the initial block. To find the best of these designs (which all have maximum trace) we use the best cyclic incomplete block design,IBD, with block size q, obtainable e.g. from the package CycDesigN (Whitaker et al. 2008). Then the

columns are organized so that the column design is as balanced as possible, which is easily done with help of a computer.
For example with v,v,p,q = 7,7,2,4 there is a BIBD with 7 treatments and block size 4. It has initial block 1 2 3 5 and the best initial block for the row-column design is easily seen to be:

1 2 3 5
2 3 5 1

If this is denoted bNRC7.7.2.4 we could try to find a cyclic bEBRC7.7.2.4 i.e. a design with block replications a=1 and a+1=2 (instead of 0 and p=2). Now we have first to find the best IBD with block size d=1 which is easy. In fact we quickly obtain a design with initial block:

1 2 3 5
6 7 4 1

which has block and row but not column balance.
How good are these designs compared to balanced designs if they existed ? Using the expression for A_4 in section 3 we can compute APV and thus compare unbalanced and (imagined) balanced designs. It is seen below that we may lose little.
In this way designs are found for those v,p,q in Tab 1 that has b > v. Their initial blocks are presented in Appendix C and some characteristics in Tab. 2.

Table 2: Partial balances and efficiencies for some cyclic unbalanced designs, bNRC and bEBRC. Efficiencies are measured relative imagined, here non-existing, balanced designs BNRC and BEBRC. Also given are efficiencies of non-existing BEBRC relative BNRC.

	bNRC		bEBRC		BEBRC
v,p,q	part. bal. for	eff.	part. bal. for	eff.	eff.
5,2,3	-	.98	block,row	.96	.78
5,3,3	-	.98	block	.99	.81
6,2,4	-	.98	-	.98	.88
6,3,3	-	.98	-	.99	.78
6,3,4	-	.99	block	.98	.88
6,4,4	-	.99	-	.99	.90
7,2,4	block	.98	block,row	.97	.81
7,2,5	-	.99	block	.97	.92
7,4,5	-	.997	block, col	.999	.93

6. Discussion

By looking at $T = \Sigma_i c_i/w_i$ on p 3 we can see which design of BNRC and BEBRC is the best. If we then let some variance components tend to infinity (or zero) various cases of interest can be covered.

When rows and columns are fixed $T = c_4/w_4$ and then BNRC has smallest variance. This is true whether blocks are fixed or random.

If we ignore rows and/or columns and let the others be random BEBRC is the better alternative.
When one of rows and columns is ignored and the other considered fixed the designs give the same pairwise variances (c_4 and c_3+c_4 for BNRC equal c_2+c_4 and c_3+c_4 respectively for BEBRC).

In practice the variance components are seldom known but instead estimated e.g. by proc Mixed in the SAS-package. For small experiments the conclusions for random models with known variance components may then be changed. Extensive (50 000 or 100 000 runs per case) simulations indicate (see Table 3) that for reasonably large experiments we can with confidence use proc Mixed. A design with (v,b,p,q)=(3,3,2,2) should of course never be used as it has 2, 3, 3 and 1 df for the four sums of squares. However, for BEBRC to be better than BNRC the variance between blocks has to dominate the variances between rows and between columns a bit more than with known variance components. They are estimated with better precision in BNRC as then treatments are orthogonal to rows and when p=q also to columns.

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Table 3. Standard errors for estimated pairwise contrasts in NRCDs with v treatments, b blocks, p rows per block, q columns per block and variance components $\sigma_b^2 = .64$, σ_{br}^2, σ_{bc}^2, $\sigma^2 = 1$. Both known and estimated variance components are used.

Designvbpq	Fix model	Random model with $\sigma_{br}^2 = .01$, $\sigma_{bc}^2 = .04$			$\sigma_{br}^2 = .04$, $\sigma_{bc}^2 = .16$			$\sigma_{br}^2 = .25$, $\sigma_{bc}^2 = .25$		
		known	estim	change	known	estim	change	known	estim	change
BNRC3322	.8165	.7817	.8043	+2.9%	.7842	.8054	+2.7%	.7882	.8076	+2.5%
BEBRC3322	.9428	.7305	.8014	+9.7%	.7479	.8151	+9.0%	.7779	.8393	+7.9%
BNRC4433	.5	.4955	.4984	+.6%	.4958	.4977	+.4%	.4963	.4978	+.3%
BEBRC4433	.5303	.4772	.4840	+1.4%	.4845	.4919	+1.5%	.4957	.5014	+1.1%
BNRC51023	.5164	.4424	.4494	+1.6%	.4524	.4613	+2.0%	.4590	.4660	+1.5%
BEBRC51023	.5855	.4193	.4291	+2.3%	.4356	.4456	+2.3%	.4560	.4676	+2.5%
BNRC5544	.3651	.3641	.3649	+.2%	.3641	.3646	+.1%	.3642	.3645	+.1%
BEBRC5544	.3771	.3560	.3583	+.6%	.3598	.3616	+.5%	.3651	.3670	+.5%

Appendix

A. Initial blocks for cyclic designs.

v,p,q	BNRC		BEBRC	
3,2,2	1 2		1 2	
	2 1		3 1	
4,2,3	1 2 3		1 2 3	1 2 3 (partial set)
	2 3 1		4 1 2	3 4 1
4,3,3	1 2 3		1 2 3	
	2 3 1		2 1 4	
	3 1 2		4 3 1	
5,2,3	1 2 3	1 2 4	1 2 3	1 2 4
	2 3 1	4 1 2	4 1 5	3 1 5
5,2,4	1 2 3 4		1 2 3 4	1 3 2 4
	3 1 4 2		5 1 2 3	4 1 5 2
5,3,3	1 2 3	1 2 4	1 2 3	1 2 4
	2 3 1	2 4 1	4 5 1	2 3 5
	3 1 2	4 1 2	2 3 4	5 1 3
5,3,4	1 2 3 4		1 2 3 4	1 3 2 4
	2 3 4 1		2 3 5 1	3 2 5 1
	3 4 1 2		4 1 2 5	4 1 3 5
5,4,4	1 2 3 4		1 2 3 4	
	2 3 4 1		5 1 2 3	
	3 4 1 2		4 5 1 2	
	4 1 2 3		3 4 5 1	
6,3,4	1 2 3 4	1 2 4 6	Non-cyclic	
	2 3 4 1	2 4 6 1		
	3 4 1 2	4 6 1 2		
	1 3 4 6 (part. set)			
	3 4 6 1			
	4 6 1 3			

v,p,q	BNRC	BEBRC
6,4,4	1 2 3 4 1 2 3 5	Non-cyclic
	2 3 4 1 2 3 5 1	
	3 4 1 2 3 5 1 2	
	4 1 2 3 5 1 2 3	
	1 2 4 5 (part set)	
	2 4 5 1	
	4 5 1 2	
	5 1 2 4	
6,4,5	1 2 3 4 5	Non-cyclic
	2 3 4 5 1	
	3 4 5 1 2	
	4 5 1 2 3	
6,5,5	1 2 3 4 5	1 2 3 4 5
	2 3 4 5 1	6 1 2 3 4
	3 4 5 1 2	5 6 1 2 3
	4 5 1 2 3	4 5 6 1 2
	5 1 2 3 4	3 4 5 6 1
7,2,4	1 4 5 6 1 4 5 6	1 5 4 7 1 2 5 7
	4 5 6 1 5 6 1 4	3 2 6 1 3 6 4 1
	1 4 5 6	1 2 7 6
	6 1 4 5	5 3 4 1
7,2,5	1 2 3 4 5	1 3 4 5 7
	2 1 4 5 3	2 1 6 3 4
	1 2 3 4 6	1 3 4 2 5
	3 4 6 1 2	6 7 1 3 4
	1 3 4 5 7	1 3 4 2 7
	5 1 3 7 4	3 6 5 1 4
7,2,6	1 2 3 4 5 6	1 2 3 4 5 6
	2 4 5 1 6 3	4 7 1 3 2 5
		1 2 3 4 6 7
		6 5 4 2 3 1
		1 2 3 5 6 7
		2 6 1 4 3 5

v,p,q	BNRC	BEBRC		
7,3,3	1 2 4	1 5 4	1 3 7	1 2 5
	2 4 1	2 4 7	3 4 5	6 7 2
	4 1 2	3 6 1	6 2 1	4 3 1
7,3,4	1 2 3 5	1 2 3 5	1 2 3 5	1 4 6 5
	2 3 5 1	4 7 2 1	4 7 6 1	2 1 5 7
	3 5 1 2	3 6 5 7	2 3 4 7	3 6 2 4
7,3,5	1 2 4 5 7	1 2 3 4 7		
	2 7 5 4 1	5 1 2 6 4		
	4 5 1 7 2	6 7 5 1 3		
7,3,6	1 2 3 4 5 6	1 3 6 7 2 4		
	3 4 5 1 6 2	6 7 1 5 3 2		
	5 1 6 2 4 3	7 6 3 4 5 1		
7,4,4	1 2 3 5	1 2 3 5	1 2 3 5	1 2 3 5
	2 3 5 1	4 7 1 6	4 6 7 1	7 1 4 6
	3 5 1 2	6 3 4 2	2 1 6 3	4 6 5 1
	5 1 2 3	7 5 2 1	7 3 5 4	2 3 7 4
7,4,5	1 2 3 4 5	1 2 3 4 5		
	3 5 4 2 1	6 7 1 2 3		
	2 3 5 1 4	4 5 6 7 1		
	5 4 1 3 2	5 6 2 3 4		
	1 2 3 7 5	1 2 3 4 6		
	2 1 7 5 3	5 7 2 1 3		
	3 7 5 2 1	4 5 7 6 2		
	7 5 1 3 2	3 6 4 5 1		
	1 2 3 6 5	1 2 3 6 5		
	2 1 6 5 3	7 1 5 4 3		
	3 6 5 2 1	2 4 7 3 6		
	6 5 1 3 2	6 5 4 2 1		
7,4,6	1 2 3 4 5 6	1 2 3 4 5 6		
	2 3 4 5 6 1	6 1 5 7 2 3		
	3 4 5 6 1 2	7 6 4 2 1 5		
	5 6 1 2 4 3	5 4 1 3 6 7		

v,p,q	BNRC	BEBRC		
7,5,6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6
	2 3 4 5 6 1	5 3 4 7 2 1	3 1 7 5 4 2	4 1 5 2 3 7
	3 4 5 6 1 2	2 6 1 3 4 7	2 3 6 7 1 4	3 7 4 6 1 2
	4 5 6 1 2 3	6 1 5 2 7 3	5 7 2 3 6 1	6 5 2 7 4 1
	5 6 1 2 3 4	4 7 6 5 1 2	4 6 5 1 3 7	7 4 1 5 6 3
7,6,6	1 2 3 4 5 6	1 2 3 4 5 6		
	2 3 4 5 6 1	7 1 2 3 4 5		
	3 4 5 6 1 2	6 7 1 2 3 4		
	4 5 6 1 2 3	5 6 7 1 2 3		
	5 6 1 2 3 4	4 5 6 7 1 2		
	6 1 2 3 4 5	3 4 5 6 7 1		

B. Non-cyclic designs

BNRC6.15.2.4

1 2 3 4	1 2 3 5	1 2 3 6	1 2 4 5	1 2 4 6
2 3 4 1	3 5 1 2	6 1 2 3	5 1 2 4	4 6 1 2
1 2 5 6	1 3 4 5	1 3 4 6	1 3 5 6	1 4 5 6
2 5 6 1	5 1 3 4	3 4 6 1	5 6 1 3	4 5 6 1
2 3 4 5	2 3 4 6	2 3 5 6	2 4 5 6	3 4 5 6
4 5 2 3	3 4 6 2	3 5 6 2	4 6 2 5	5 6 4 3

BEBRC6.15.2.4

4 6 1 2	3 5 6 1	5 1 2 4	1 3 4 5	6 4 5 1
5 1 2 3	1 2 3 4	1 3 4 6	2 5 1 6	3 2 1 6
5 2 3 1	2 4 5 6	4 2 3 5	4 1 2 6	6 3 4 2
2 6 4 3	3 1 4 2	2 1 5 6	6 3 5 2	1 4 5 3
3 6 1 5	5 3 4 6	5 6 4 3	2 5 6 4	6 1 5 4
5 3 2 4	1 6 3 2	1 4 2 5	3 6 4 1	5 6 2 3

BNRC6.6.2.5

1 2 3 4 5	1 2 3 4 6	1 2 3 5 6	1 2 4 5 6	1 3 4 5 6	2 3 4 5 6
2 3 4 5 1	3 4 6 1 2	5 6 1 2 3	6 1 2 4 5	4 5 6 3 1	3 4 6 2 5

BEBRC6.15.2.5

1 2 3 4 6	1 2 3 5 6	1 2 3 4 6	1 2 4 5 6	1 2 3 4 6
5 1 2 3 4	4 5 1 2 3	3 6 2 5 1	2 4 1 3 5	5 6 4 2 1

1 2 4 5 6	1 3 4 5 6	1 3 4 5 6	1 3 4 5 6	1 3 4 5 6
2 5 6 3 1	5 1 2 4 3	4 2 3 6 1	2 5 1 6 3	5 1 6 4 2

2 3 4 5 6	2 3 4 5 6	2 3 4 5 6	2 5 6 3 4	1 3 4 6 5
1 2 3 4 5	4 1 6 2 3	6 5 1 2 3	4 1 2 5 6	6 2 3 5 4

BNRC6.10.3.3

3*3 Latin squares with treatments

(1,3,4) (1,5,6) (2,3,6) (1,3,6) (3,4,5) (2,4,6) (1,2,5) (4,5,6) (1,2,4)

BEBRC6.10.3.3

See p 5 above.

BEBRC6.5.3.4

1 3 4 5	1 3 4 6	1 2 3 4	1 2 4 5	1 2 3 5
6 4 2 1	5 6 1 2	3 1 5 6	6 3 1 2	6 5 1 4
5 2 6 3	2 4 5 3	2 4 6 5	4 5 3 6	2 4 6 3

BNRC6.6.3.5

1 2 3 4 5	2 3 4 5 6	3 4 5 6 1	4 5 6 1 2	5 6 1 2 3	6 1 2 3 4
2 3 4 5 1	3 4 5 6 2	5 6 1 3 4	1 2 4 5 6	3 5 6 1 2	4 2 1 6 3
4 1 5 2 3	4 5 6 2 3	4 1 6 5 3	5 6 1 2 4	2 1 3 5 6	2 3 4 1 6

BEBRC6.10.3.5

3 4 5 1 2	3 5 6 1 2	1 4 6 2 3	1 3 6 2 4	1 2 3 4 5
5 3 1 6 4	4 2 3 5 6	3 5 1 4 6	5 2 1 3 6	4 3 2 1 6
2 6 4 5 3	6 1 4 3 5	2 6 4 1 5	3 6 4 1 5	3 1 6 5 2

1 4 5 2 3	1 2 5 4 6	2 3 4 1 5	2 5 6 3 1	2 4 6 5 3
6 2 1 4 5	6 3 2 5 1	1 2 3 4 6	4 3 2 5 6	1 5 4 6 2
4 5 3 1 6	2 5 4 1 3	6 4 5 3 2	5 6 4 1 2	6 2 3 1 4

BEBRC6.15.4.4

2 1 5 3	3 1 4 2	4 6 2 1	2 6 5 3	3 1 5 4
3 2 6 4	1 2 6 3	5 2 4 3	4 1 6 2	5 2 6 3
5 3 1 6	6 5 1 4	1 3 5 4	3 2 1 4	4 3 1 6
6 4 2 5	4 3 5 6	2 1 6 5	6 4 3 5	1 4 2 5

4 1 5 2	6 1 4 3	6 1 2 4	1 3 2 4	1 5 3 2
5 2 6 4	1 3 2 5	1 2 6 3	2 4 5 3	2 1 6 3
6 3 1 5	3 4 5 6	2 3 4 5	3 6 1 5	3 4 1 6
2 6 4 3	5 6 1 2	4 5 1 6	5 1 6 2	6 2 5 4

2 1 3 4	3 1 5 2	4 1 5 6	5 6 1 2	6 5 1 4
3 5 6 2	4 2 1 5	5 4 3 2	6 1 3 4	1 2 3 6
4 2 1 6	5 3 4 6	6 2 4 3	1 3 4 5	2 3 6 5
1 4 5 3	2 4 6 3	3 6 1 5	4 2 5 6	5 1 4 2

BEBRC6.15.4.5

Nine of the 15 blocks can be obtained by cycling the initial blocks

1 2 3 4 5	1 2 3 4 5	(part. set)
3 4 2 6 1	4 6 2 3 1	
6 3 1 5 2	3 1 4 5 6	
2 6 5 1 4	6 4 5 1 2	

Six further blocks are as follows:

1 3 2 4 5	2 4 6 1 3	3 5 4 1 6	4 6 2 1 3	1 5 3 6 4	6 2 3 1 5
6 1 3 2 4	1 2 4 3 5	1 3 5 2 4	1 4 6 2 5	2 1 5 3 6	1 6 2 3 4

5 6 4 1 3	6 5 1 2 4	4 6 3 5 2	5 2 3 4 6	3 4 2 1 5	4 1 5 6 2
3 2 1 5 6	4 6 3 5 2	5 2 1 6 3	6 3 1 5 4	4 6 1 5 2	2 5 6 4 3

C. Initial blocks for non-balanced designs.

v,p,q	bNRC	bEBRC
5,2,3	1 2 5	1 3 4
	5 1 2	2 1 5
5,3,3	1 2 3	1 2 3
	2 3 1	4 3 1
	3 1 2	2 4 5
6,2,4	1 2 3 4	1 2 3 4
	4 3 1 2	6 3 5 1
6,3,3	1 2 4	1 2 3
	2 4 1	4 6 5
	4 1 2	6 1 4
6,3,4	1 2 3 4	1 2 3 4
	4 1 2 3	5 6 1 2
	3 4 1 2	4 3 6 5
6,4,4	1 2 3 4	1 2 3 4
	2 3 4 1	2 5 6 1
	4 1 2 3	3 4 1 5
7,2,4	1 2 3 5	1 2 3 5
	2 3 5 1	6 7 4 1
7,2,5	1 2 3 4 5	1 2 3 4 5
	4 3 5 2 1	6 7 2 1 4
7,4,5	1 2 3 4 5	1 2 3 4 5
	2 3 4 5 1	6 3 1 2 7
	3 4 5 1 2	2 5 6 7 4
	4 5 1 2 3	5 1 4 3 6

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