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# Exact and superlative measurement of the Luenberger-Hicks-Moorsteen productivity indicator

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# EXACT AND SUPERLATIVE MEASUREMENT OF THE LUENBERGER-HICKS-MOORSTEEN PRODUCTIVITY INDICATOR

FREDERIC ANG AND PIETER JAN KERSTENS

ABSTRACT. This paper shows that the Bennet-Bowley profit indicator is an exact and superlative approximation of the additively complete Luenberger-Hicks-Moorsteen productivity indicator when the input and output directional distance functions can be represented up to the second order by a quadratic functional form. It also establishes the conditions under which the exact and superlative measures of the Luenberger productivity indicator and Luenberger-Hicks-Moorsteen productivity indicator coincide.
**Keywords:** Bennet-Bowley, difference-based indicator, total factor productivity, exact and superlative approximation.
**JEL classification:** C43, D21, D24.

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## 1. Introduction

Productivity analysis is an essential tool to monitor economic performance. The index number approach to productivity measurement depends on quantities and prices. The distance function approach allows analyzing the sources of productivity change with various decompositions. For instance, productivity growth is in this approach commonly decomposed into changes in distances to the technological frontier (efficiency change) and changes of the production technology (technical change). However, the distance function approach relies on the estimation of production functions. Since computation of index numbers is straightforward and the underlying production technology is inherently unknown, establishing relationships between index numbers and distance functions is the focus of an important body of literature. Index numbers approximate distance-function-based productivity measures in an *exact* way if one can assume (i) economic optimizing behavior and (ii) a certain functional form of the technology. A *superlative* approximation weakens (ii) by only assuming that the technology can be approximated up to the second order by a flexible functional form (Diewert, 1976).

The literature has largely focused on exact and superlative approximation of *ratio*-based "indexes". Caves et al. (1982) show that the Törnqvist index is a superlative index of the Malmquist index under certain optimizing behavior and a translog functional form. This result also holds for the Fisher ideal index (Balk, 1993; Färe and Grosskopf, 1992). Diewert and Fox (2010) suggest that the Törnqvist index is also an exact and superlative approximation of Bjurek (1996)'s Hicks-Moorsteen index without any returns-to-scale assumption. Unlike the Malmquist index, the Hicks-Moorsteen index is "multiplicatively complete" in that it consists of a ratio of an output aggregator to an input aggregator (O'Donnell, 2012). This allows for disentangling the exact contribution of output and input change to productivity. Mizobuchi (2016) shows that the Törnqvist index is an exact and superlative approximation of both the Malmquist index and Hicks-Moorsteen index under constant-returns-to-scale (Proposition 4, p.11). Furthermore, he demonstrates that this assumption can be loosened to $\alpha$-returns-to-scale for the Hicks-Moorsteen index, but *not* for the Malmquist index (Proposition 5, p.14).

Ratio-based indexes can be undefined when zeros occur in the numerator or denominator and are not translation-invariant. *Difference*-based "indicators" such as Chambers et al. (1996)'s Luenberger productivity indicator avoid these drawbacks altogether (Fox, 2006).[1] Furthermore, differences are more common in, for example, accounting. We refer to Diewert (2005) for a systematic comparison of the ratio and difference approaches to index numbers from a test and economic perspective. Chambers (2002) introduces the Bennet-Bowley profit measure and establishes that it is an exact and superlative approximation of the Luenberger

[1] We follow Diewert (2005) in referring to productivity measures based on differences as "indicators".

productivity indicator when it can be represented up to the second order by a quadratic functional form.

However, the Luenberger productivity indicator is not "additively complete", preventing decomposition into output and input changes (O'Donnell, 2012). Although Briec and Kerstens (2004)'s Luenberger-Hicks-Moorsteen (LHM) productivity indicator does have this desirable property (Ang and Kerstens, 2016), no superlative and exact approximation is currently known in the literature.

This paper shows that the Bennet-Bowley profit indicator is an exact and superlative approximation of Briec and Kerstens (2004)'s additively complete LHM productivity indicator when the input and output directional distance functions can be represented up to the second order by a quadratic functional form. The Bennet-Bowley profit indicator thus approximates both Chambers (2002)'s incomplete Luenberger productivity indicator and Briec and Kerstens (2004)'s additively complete LHM productivity indicator under equivalent conditions. This parallels Mizobuchi (2016)'s finding that the Törnqvist index is an exact and superlative approximation of both the multiplicatively complete Hicks-Moorsteen index and incomplete Malmquist productivity index under equivalent conditions.

This paper is structured as follows. We first introduce necessary notation and definitions of the Luenberger and Bennet-Bowley indicators. We then define the LHM indicator and present our main results before concluding.

## 2. Linking the Luenberger productivity indicator to the Bennet-Bowley profit indicator

This section sets the stage for our main result by introducing necessary notation and definitions of the Luenberger productivity indicator and the Bennet-Bowley cost, revenue and profit indicators. It also reminds the reader of Chambers (2002)'s result which links the Bennet-Bowley profit indicator as an exact and superlative approximation of the Luenberger productivity indicator.

### 2.1. The Luenberger productivity indicator.

Let $\mathbf{x}_t \in \mathbb{R}^n_+$ be the inputs that are used to produce outputs $\mathbf{y}_t \in \mathbb{R}^m_+$. We define the production possibility set as:

$$\boldsymbol{\mathcal{T}}_t = \left\{ (\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}^{n+m}_+ | \mathbf{x}_t \text{ can produce } \mathbf{y}_t \right\}.$$

We make the following assumptions on the production possibility set (Chambers, 2002):

**Axiom 1** (Closedness). *$\boldsymbol{\mathcal{T}}_t$ is closed.*

**Axiom 2** (Free disposability of inputs and outputs). *if $(\mathbf{x}'_t, -\mathbf{y}'_t) \geq (\mathbf{x}_t, -\mathbf{y}_t)$ then $(\mathbf{x}_t, \mathbf{y}_t) \in \boldsymbol{\mathcal{T}}_t \Rightarrow (\mathbf{x}'_t, \mathbf{x}'_t) \in \boldsymbol{\mathcal{T}}_t$.*

**Axiom 3** (Inaction). *Inaction is possible: $(\mathbf{0}^n, \mathbf{0}^m) \in \boldsymbol{\mathcal{T}}_t$.*

The directional distance function was first introduced in a production context by Chambers et al. (1996). We denote the time-related directional distance function for $(a,b) \in \{t, t+1\} \times \{t, t+1\}$:

$$D_b(\mathbf{x}_a, \mathbf{y}_a; \mathbf{g}_a) = \sup\left\{\beta \in \mathbb{R} : (\mathbf{x}_a - \beta \mathbf{g}_a^i, \mathbf{y}_a + \beta \mathbf{g}_a^o) \in \boldsymbol{\mathcal{T}}_b\right\}, \tag{1}$$

if $(\mathbf{x}_a - \beta \mathbf{g}_a^i, \mathbf{y}_a + \beta \mathbf{g}_a^o) \in \boldsymbol{\mathcal{T}}_b$ for some $\beta$ and $D_b(\mathbf{x}_a, \mathbf{y}_a; \mathbf{g}_a) = -\infty$ otherwise. Here, $\mathbf{g}_a = (\mathbf{g}_a^i, \mathbf{g}_a^o)$ represents the directional vector.

Chambers (2002) defines the Luenberger productivity indicator as:

$$\begin{aligned} &L_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t, \mathbf{g}_{t+1}) \\ &\quad = \frac{1}{2}\left[(D_t(\mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t) - D_t(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_{t+1}))\right. \\ &\quad \left. + (D_{t+1}(\mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t) - D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_{t+1}))\right]. \end{aligned} \tag{2}$$

It can be decomposed in technical change and technical inefficiency change (Chambers et al., 1996), but the exact contribution of output and input change cannot be determined. This is because, in general, $\mathbf{g}_t = (\mathbf{g}_t^i, \mathbf{g}_t^o) > 0$ and inputs are contracted simultaneously as outputs are expanded in the directional distance functions (Ang and Kerstens, 2016). Hence, it is not “additively complete” (O’Donnell, 2012). Furthermore, unlike the Luenberger-Hicks-Moorsteen (Briec and Kerstens, 2011), the Luenberger productivity indicator is not “determinate” in that it can be undefined (Briec and Kerstens, 2009).[2]

Furthermore, Chambers (2002) defines the output-quantity Luenberger productivity indicator as:

$$LO_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t^o, \mathbf{g}_{t+1}^o) = \frac{1}{2}\left[LO_t + LO_{t+1}\right] \tag{3a}$$

where the base period $t$ output-quantity indicator is defined as:

$$LO_t(\mathbf{x}_t, \mathbf{y}_t, \mathbf{y}_{t+1}; \mathbf{g}_t^o, \mathbf{g}_{t+1}^o) = D_t(\mathbf{x}_t, \mathbf{y}_t; (0, \mathbf{g}_t^o)) - D_t(\mathbf{x}_t, \mathbf{y}_{t+1}; (0, \mathbf{g}_{t+1}^o)), \tag{3b}$$

and the base period $t+1$ output-quantity indicator:

$$LO_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{y}_t; \mathbf{g}_t^o, \mathbf{g}_{t+1}^o) = D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_t; (0, \mathbf{g}_t^o)) - D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; (0, \mathbf{g}_{t+1}^o)) \tag{3c}$$

The input-quantity Luenberger productivity indicator:

$$LI_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t^i, \mathbf{g}_{t+1}^i) = \frac{1}{2}\left[LI_t + LI_{t+1}\right] \tag{4a}$$

where the base period $t$ input-quantity indicator is defined as:

$$LI_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t; \mathbf{g}_t^i, \mathbf{g}_{t+1}^i) = D_t(\mathbf{x}_t, \mathbf{y}_t; (\mathbf{g}_t^i, 0)) - D_t(\mathbf{x}_{t+1}, \mathbf{y}_t; (\mathbf{g}_{t+1}^i, 0)) \tag{4b}$$

[2]Determinateness of the index does not necessarily carry over to its components. This occurs for example in the empirical application of Ang and Kerstens (2016), where the technical change component is undefined for one of the observations.

and the base period $t+1$ input-quantity indicator:

(4c)
$$LI_{t+1}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t^i, \mathbf{g}_{t+1}^i) = D_{t+1}(\mathbf{x}_t, \mathbf{y}_{t+1}; (\mathbf{g}_t^i, 0)) - D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; (\mathbf{g}_{t+1}^i, 0))$$

The output-quantity (input-quantity) Luenberger productivity indicator $LO_{t,t+1}(\cdot)$ ($LI_{t,t+1}(\cdot)$) measures productivity solely in the output (input) directions. Thus, a combination of both $LO_{t,t+1}(\cdot)$ and $LI_{t,t+1}(\cdot)$ is "additively complete". This is Briec and Kerstens (2004)'s Luenberger-Hicks-Moorsteen productivity indicator (see Section 3 *infra*).

2.2. **Bennet-Bowley indicators.** The preceding productivity measures have the advantage that they can be computed in the absence of price data, but their major drawback is that they require the approximation of the technology set and estimation of distance functions. This makes them somewhat harder to compute. We now focus our attention to productivity measures that are easy to compute using price data and which do not require estimation of distance functions.

Assume that the preceding distance functions can be estimated by a quadratic functional form:

$$\begin{aligned} D_h(\mathbf{x}, \mathbf{y}; (\mathbf{g}^i, \mathbf{g}^o)) = a_0^h + \sum_{u=1}^{n} a_u^h x_u + \sum_{k=1}^{m} b_k^h y_k + \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n} \alpha_{uv}^h x_u x_v \\ + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} \beta_{kl}^h y_k y_l + \frac{1}{2}\sum_{u=1}^{n}\sum_{k=1}^{m} \gamma_{uk}^h x_u y_k, \end{aligned} \tag{5a}$$

with the restrictions

$$\alpha_{uv}^h = \alpha_{vu}^h, \beta_{kl}^h = \beta_{lk}^h, \tag{5b}$$

$$\sum_{k=1}^{m} b_k^h g_k^o - \sum_{u=1}^{n} a_u^h g_u^i = -1; \tag{5c}$$

$$\sum_{k=1}^{m} \gamma_{uk}^h g_k^o - \sum_{v=1}^{n} \alpha_{uv}^h g_v^i = 0, \qquad u = 1, \dots, n; \tag{5d}$$

$$\sum_{l=1}^{m} \beta_{kl}^h g_l^o - \sum_{u=1}^{n} \gamma_{uk}^h g_u^i = 0, \qquad k = 1, \dots, m. \tag{5e}$$

The output (input) directional distance function is defined by setting $\mathbf{g}^{i(o)} = \mathbf{0}^{n(m)}$.

Chambers (2002) then defines the Bennet-Bowley profit indicator (6a) by the difference between the Bennet-Bowley revenue indicator (6b) and Bennet-Bowley cost indicator (6c):

$$BP(\mathbf{p}_t, \mathbf{p}_{t+1}, \mathbf{w}_t, \mathbf{w}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}) = BR(\mathbf{p}_t, \mathbf{p}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) - BC(\mathbf{w}_t, \mathbf{w}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}). \tag{6a}$$

with

$$BR(\mathbf{p}_t, \mathbf{p}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \frac{1}{2}\left[\mathbf{p}_t(\mathbf{y}_{t+1} - \mathbf{y}_t) + \mathbf{p}_{t+1}(\mathbf{y}_{t+1} - \mathbf{y}_t)\right], \tag{6b}$$

and

$$BC(\mathbf{w}_t, \mathbf{w}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}) = \frac{1}{2}\left[\mathbf{w}_t(\mathbf{x}_{t+1} - \mathbf{x}_t) + \mathbf{w}_{t+1}(\mathbf{x}_{t+1} - \mathbf{x}_t)\right]. \tag{6c}$$

Avoiding any estimation procedure, these Bennet-Bowley indicators are straightforward to compute from available data. Hence, it is of practical interest to establish the conditions under which the Luenberger productivity indicator can be computed by a Bennet-Bowley profit indicator.

**Proposition 1** (Theorem 6 in Chambers (2002))**.** *If firms maximize profit, and the technology directional distance function is quadratic with $\alpha_{ij}^t = \alpha_{ij}^{t+1}$ for all $i$ and $j$, $\beta_{ij}^t = \beta_{ij}^{t+1}$ for all $i$ and $j$ then*

$$L_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t, \mathbf{g}_{t+1}) = BP(\hat{\mathbf{p}}_t, \hat{\mathbf{p}}_{t+1}, \hat{\mathbf{w}}_t, \hat{\mathbf{w}}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$$

*where* $\hat{\mathbf{p}}_k = \frac{\mathbf{p}_k}{\mathbf{p}_k\mathbf{g}_k^o + \mathbf{w}_k\mathbf{g}_k^i}$ *and* $\hat{\mathbf{w}}_k = \frac{\mathbf{w}_k}{\mathbf{p}_k\mathbf{g}_k^o + \mathbf{w}_k\mathbf{g}_k^i}$.

It turns out that the Bennet-Bowley profit indicator is an exact and superlative indicator of the Luenberger productivity indicator under an appropriate price normalization and when the directional distance function can be approximated by the quadratic functional form (5).

## 3. Exact and superlative measurement of Luenberger-Hicks-Moorsteen productivity

Briec and Kerstens (2004) define the LHM productivity indicator with base period $t$ as the difference between the Luenberger output quantity indicator and the Luenberger input quantity indicator:[3]

$$\begin{aligned} &LHM_t(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t, \mathbf{g}_{t+1}) \\ &= (D_t(\mathbf{x}_t, \mathbf{y}_t; (0, \mathbf{g}_t^o)) - D_t(\mathbf{x}_t, \mathbf{y}_{t+1}; (0, \mathbf{g}_{t+1}^o))) \\ &- (D_t(\mathbf{x}_{t+1}, \mathbf{y}_t; (\mathbf{g}_{t+1}^i, 0)) - D_t(\mathbf{x}_t, \mathbf{y}_t; (\mathbf{g}_t^i, 0))) \\ &\equiv LO_t(\mathbf{x}_t, \mathbf{y}_t, \mathbf{y}_{t+1}; \mathbf{g}_t^o, \mathbf{g}_{t+1}^o) - [-LI_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t; \mathbf{g}_t^i, \mathbf{g}_{t+1}^i)]. \end{aligned} \tag{7}$$

[3]We follow Chambers (2002)'s definition of the input quantity indicator, swapping the places of Briec and Kerstens (2004)'s definition.

The LHM productivity indicator with base period $t+1$ is:

$$
\begin{aligned}
(8)\qquad & LHM_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t, \mathbf{g}_{t+1}) \\
&= (D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_t; (0, \mathbf{g}_t^o)) - D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; (0, \mathbf{g}_{t+1}^o))) \\
&- (D_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}; (\mathbf{g}_{t+1}^i, 0)) - D_{t+1}(\mathbf{x}_t, \mathbf{y}_{t+1}; (\mathbf{g}_t^i, 0))) \\
&\equiv LO_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{y}_t; \mathbf{g}_t^o, \mathbf{g}_{t+1}^o) + LI_{t+1}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t^i, \mathbf{g}_{t+1}^i).
\end{aligned}
$$

One takes an arithmetic mean of $LHM_t$ and $LHM_{t+1}$ to avoid an arbitrary choice of base periods:

$$
\begin{aligned}
(9)\qquad & LHM_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t, \mathbf{g}_{t+1}) \\
&= \frac{1}{2}\left[LHM_t(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t, \mathbf{g}_{t+1})\right. \\
&\left.+ LHM_{t+1}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{y}_t; \mathbf{g}_t, \mathbf{g}_{t+1})\right]
\end{aligned}
$$

Recently, Ang and Kerstens (2016) show that the LHM productivity indicator is “additively complete” and, following Diewert and Fox (2017), provide a decomposition in the usual components of technical change, technical inefficiency change and scale inefficiency change under minimal assumptions of the technology set. However, an exact and superlative approximation of the LHM productivity indicator is presently absent in the literature, which disallows easy computation of the LHM productivity indicator in practice. Our main result fills this gap:

**Proposition 2.** *If firms maximize profit and the directional distance function is quadratic with $\alpha_{ij}^t = \alpha_{ij}^{t+1}$ for all $i$ and $j$, $\beta_{ij}^t = \beta_{ij}^{t+1}$ for all $i$ and $j$ then*

$$
\begin{aligned}
LHM_{t,t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}; \mathbf{g}_t, \mathbf{g}_{t+1}) &= BP(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}) \\
&= BR(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) - BC(\mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{x}_t, \mathbf{x}_{t+1}),
\end{aligned}
$$

*where $\tilde{\mathbf{p}}_k = \frac{\mathbf{p}_k}{\mathbf{p}_k \mathbf{g}_k^o}$ and $\mathbf{w}_k^* = \frac{\mathbf{w}_k}{\mathbf{w}_k \mathbf{g}_k^i}$.*

*Proof.* We can write

$$
LHM_{t,t+1}(\cdot) = \frac{1}{2}\left[LO_t(\cdot) + LO_{t+1}(\cdot)\right] + \frac{1}{2}\left[LI_t(\cdot) + LI_{t+1}(\cdot)\right].
$$

From Theorem 4 in Chambers (2002) it follows that

$$
\frac{1}{2}\left[LO_t(\cdot) + LO_{t+1}(\cdot)\right] = BR(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})
$$

if firms maximize revenue and technology is quadratic with $\beta_{ij}^t = \beta_{ij}^{t+1}$ for all $i$ and $j$. From Theorem 2 in Chambers (2002) we know that

$$
\frac{1}{2}\left[LI_t(\cdot) + LI_{t+1}(\cdot)\right] = -BC(\mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{x}_t, \mathbf{x}_{t+1})
$$

if firms minimize costs and technology is quadratic with $\alpha_{ij}^t = \alpha_{ij}^{t+1}$ for all $i$ and $j$. Simultaneous revenue maximization and cost minimization is profit maximization, which yields the desired result. □

The condition under which both Bennet-Bowley profit indicators (cfr. $BP(\cdot)$ in Proposition 1 and Proposition 2) are equivalent follows immediately:

**Corollary 1.** *If firms are profit-maximizing and the directional distance function is quadratic with $\alpha_{ij}^t = \alpha_{ij}^{t+1}$ for all $i$ and $j$, $\beta_{ij}^t = \beta_{ij}^{t+1}$ for all $i$ and $j$, then*

$$BP(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$$

*with* $(\mathbf{g}_k^i, \mathbf{g}_k^o) = (\frac{\tau}{n\mathbf{w}_k}, \frac{\tau}{m\mathbf{p}_k})$ *and*

$$BP(\hat{\mathbf{p}}_t, \hat{\mathbf{p}}_{t+1}, \hat{\mathbf{w}}_t, \hat{\mathbf{w}}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$$

*with* $(\mathbf{g}_k^i, \mathbf{g}_k^o) = (\frac{\tau}{2n\mathbf{w}_k}, \frac{\tau}{2m\mathbf{p}_k})$ *coincide for any* $\tau \in \mathbb{R}$.

*Proof.* When firms maximize profit and the directional distance function is quadratic with $\alpha_{ij}^t = \alpha_{ij}^{t+1}$ for all $i$ and $j$, $\beta_{ij}^t = \beta_{ij}^{t+1}$ for all $i$ and $j$, then both Bennet-Bowley profit indicators only differ by their price normalization which is parametrized by the direction vectors. Both price normalizations coincide when:

$$\begin{aligned} &(\tilde{\mathbf{p}}_k, \mathbf{w}_k^*) = (\hat{\mathbf{p}}_k, \hat{\mathbf{w}}_k) \\ \Leftrightarrow\ &(\frac{\mathbf{p}_k}{\mathbf{p}_k\mathbf{g}_k^o}, \frac{\mathbf{w}_k}{\mathbf{w}_k\mathbf{g}_k^i}) = (\frac{\mathbf{p}_k}{\mathbf{p}_k\mathbf{g}_k^o + \mathbf{w}_k\mathbf{g}_k^i}, \frac{\mathbf{w}_k}{\mathbf{p}_k\mathbf{g}_k^o + \mathbf{w}_k\mathbf{g}_k^i}) \end{aligned}$$

or when both denominators equal some $\tau \in \mathbb{R}$. For the LHS this holds when $(\mathbf{g}_k^i, \mathbf{g}_k^o) = (\frac{\tau}{n\mathbf{w}_k}, \frac{\tau}{m\mathbf{p}_k})$ and for the RHS this holds when $(\mathbf{g}_k^i, \mathbf{g}_k^o) = (\frac{\tau}{2n\mathbf{w}_k}, \frac{\tau}{2m\mathbf{p}_k})$. □

Briec and Kerstens (2004) show that if and only if the technology is (i) inversely translation homothetic in the direction of $\mathbf{g}$; and (ii) exhibits graph translation homotheticity in the direction of $\mathbf{g}$, then the LHM productivity indicator and the Luenberger output (input) productivity indicator coincide. An equivalent condition in terms of Bennet-Bowley profit indicators is the following:

**Corollary 2.** *If firms are profit-maximizing and* $\mathbf{p}_k\mathbf{g}_k^o = \mathbf{w}_k\mathbf{g}_k^i$, *then*

$$BP(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$$

*and*

$$BP(\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_{t+1}, \tilde{\mathbf{w}}_t, \tilde{\mathbf{w}}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$$

*(or* $BP(\mathbf{p}_t^*, \mathbf{p}_{t+1}^*, \mathbf{w}_t^*, \mathbf{w}_{t+1}^*, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1})$*) (locally) coincide.*

*Proof.* Trivial and follows directly from Corollary 8 in Chambers (2002). □

## 4. Conclusions

This paper shows that the Bennet-Bowley profit indicator is an exact and superlative approximation of the LHM productivity indicator when the input and output directional distance functions are quadratic in inputs and outputs. The Bennet-Bowley profit indicator thus approximates both Chambers (2002)'s incomplete Luenberger productivity indicator and Briec and Kerstens (2004)'s additively complete LHM productivity indicator under equivalent conditions (albeit for a different price normalization). This parallels Mizobuchi (2016)'s finding that the Törnqvist index is an exact and superlative approximation of both the multiplicatively complete Hicks-Moorsteen index and incomplete Malmquist productivity index under equivalent conditions. Our finding differs subtly in that it requires a different price normalization, which is not required for Mizobuchi (2016)'s finding. This is a direct consequence of the fact that ratio-based indexes, unlike difference-based indicators, are unit-invariant.

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