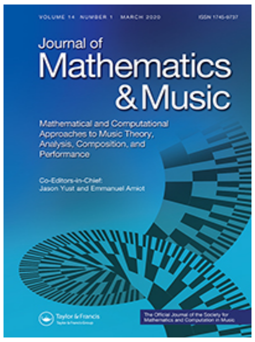

Taylor & Francis
Taylor & Francis Group

Journal of Mathematics and Music

Mathematical and Computational Approaches to Music Theory, Analysis, Composition and Performance

ISSN: 1745-9737 (Print) 1745-9745 (Online) Journal homepage: https://www.tandfonline.com/loi/tmam20

On features of fugue subjects. A comparison of J.S. Bach and later composers

Jesper Rydén

To cite this article: Jesper Rydén (2020) On features of fugue subjects. A comparison of J.S. Bach and later composers, Journal of Mathematics and Music, 14:1, 1-20, DOI: 10.1080/17459737.2019.1610193

To link to this article: https://doi.org/10.1080/17459737.2019.1610193

Published online: 17 May 2019.

Journal of Mathematics and Music, 2020
Vol. 14, No. 1, 1–20, https://doi.org/10.1080/17459737.2019.1610193

Taylor & Francis
Taylor & Francis Group

Check for updates

On features of fugue subjects. A comparison of J.S. Bach and later composers

Jesper Rydén*

Department of Energy and Technology, Swedish University of Agricultural Sciences, Uppsala, Sweden

(*Received 10 May 2017; accepted 17 April 2019*)

The musical form fugue has inspired many composers, in particular writing for the organ. By quantifying a fugue subject, comparisons can be made on a statistical basis between J.S. Bach and composers from later epochs, a priori dividing works into three categories. The quantification is made by studying the following features: length, expressed in number of notes written; range (in semitones); number of pitch classes; initial interval (in semitones); number of unique intervals between successive notes; maximum interval between successive notes (in semitones). A data set of subjects from various composers was constructed. An analysis of principal components (PCA) makes possible an interpretation of the variability as well as a visualisation of all cases. Regression models for counts are introduced to investigate differences between composers, taking into account dependence on covariates. Concerning the range of the subject, a statistically significant difference was found between Bach and other composers. Furthermore, regarding the number of unique notes employed, a statistically significant difference was found between all composer categories.

Keywords: principal components; generalised linear model; Poisson regression; fugue subject; range; pitch class

2010 Mathematics Subject Classification: 00A65; 62J12; 62H25

1. Introduction

"An essential aspect of music is structure," as pointed out by Beran (2004, vii). What is indeed fascinating with the outcomes of the art of music – that is, the compositions (or improvisations) – is the interplay between structures at various levels and the creative musical ideas creating these. For instance, in a piano sonata by Mozart, one may consider the construction of motives as well as the proportions of the sections; see Rydén (2006) for a statistical analysis of the latter problem. Mathematical and statistical techniques thus play a role for analysis of compositions; as an aim for general understanding of musical works and various composers, or for more specific purposes like classifying doubtful works (see e.g. Backer and Peter 2005) or indeed creation of new ones (Beran and Mazzola 1999).

The *fugue* is a musical art form written for a given number of voices (vocal or instrumental), see treatises by e.g. Prout (1891), Gedalge (1901), Mann (1987). Notably, a fugue is based upon one subject (or one or several countersubjects). The subject is initially heard in one part alone,

*Email: jesper.ryden@slu.se

then imitated by all other parts in turn. Typical tools from counterpoint are used for the composition: augmentation, inversion, etc. This form implies certain conditions and procedures, yet the composer has a great freedom in the creation of the movement, with respect to thematic work and general character. Note that in double fugues, two subjects are present (the second often introduced after the exposition of the first) and often combined in counterpoint at the end of the entire fugue.

In this work, we consider fugues written for the medium of the organ. The name of Johann Sebastian Bach (1685–1750) springs to mind in this context, even though some of his predecessors also wrote organ fugues, e.g. Dieterich Buxtehude or Johann Jakob Froberger. We focus on the musicological question whether certain characteristics of the fugue subject had changed over the centuries after Bach, along with more general developments in musical composition, e.g. the harmonic language, chromaticism etc. Nevertheless, it could have been the case that composers of later epochs may have regarded the fugue primarily as an ancient art form and hence written in a somewhat archaic manner, with perhaps a nod to Bach's oeuvres. One example is the final fugue of Robert Schumann's six fugues on BACH (op. 60), which has some features similar to Bach's fugue in E flat major BWV 552, which closes Clavier-Übung III, and thus could be thought of as a tribute to Bach (Stinson 2006).

A typical fugue subject possesses certain properties, from the point of view of music theory. Some of these are not easy to quantify, e.g. modulations. In this paper, the subject has been described by a set of integers, as described below. The statistical relations, or correlations, between these observations are investigated for a large number of subjects which have been compiled by the author. Works were divided into three categories. The works by Bach was considered one category, and we may also view the organ fugues of Max Reger (1873–1916) as a separate category. Harmonically, in many of Reger's fugue subjects the concept is extended from that of the predecessors, and furthermore, his output is quite large. The third category includes simply fugue subjects by other composers. The overall aim is to investigate possible differences between the categories, by exploratory statistical methodology as well as more advanced regression models where the influence on suitably chosen response variables from various covariates is investigated.

We here give an outline of the paper's organisation. In the next section, the variables describing a fugue subject are presented in more detail along with examples and discussion. In Section 3, the data set is presented and introductory data analysis is performed, in order to facilitate the later modelling. In Section 4, we study relationships between the chosen variables by analysis of principal components, with the aim of exploratory data analysis. We introduce thereafter, in Section 5, regression models for counts, and the influence on some selected variables by others can be analysed. Finally, in Section 6, a concluding discussion is given.

2. Chosen features of a fugue subject

In this paper, we describe a given fugue subject by a set of integers, $x_1, x_2, \ldots, x_6$:

x_1	length, expressed in number of notes written
x_2	range (in semitones)
x_3	number of unique pitch classes
x_4	initial interval (in semitones)
x_5	number of unique intervals between successive notes
x_6	maximum interval between successive notes (in semitones)

Figure 1. J.S. Bach: subject from Fugue in C minor (BWV 537).

Figure 2. Subject from Rheinberger's fugue in A minor, 4th organ sonata: (*Fuga cromatica*).

In addition, categorical variables are introduced in regression models for (i) the category of composer; (ii) the presence of a tritone interval between successive notes. Consider for instance, in Figure 1, the following subject by Bach, from Fantasia and Fugue in C minor (BWV 537):

In this example, we find the following observed values of the variables:

x_1	x_2	x_3	x_4	x_5	x_6
17	9	7	7	7	9

Note that the tied note in bar 2 implies one single count of the note (A flat). Next, we discuss some issues of these chosen variables.

Length, x_1. To derive the length, or in other words, define the subject of the whole composition, is not an easy task. In fact, Prout (1891) states that "It is impossible to give any definite rules as to the length of a fugue subject." Tovey (1924), comments on Bach's D major fuge from BWV 854: "It is not worthwhile settling where the subject ends and where the countersubject begins."

To find an integer representing the length of the subject, the fugue as a whole needs to be considered. How is the theme presented in the voices in the initial part of the fugue (the *exposition*), and in later entries? Composers of later epochs often write slurs in the score to indicate phrasing; however, guidance on these alone can often result in inconsequencies when the initial presentation is compared to later versions. Thus, to determine the length (and hence the subject itself) has some degree of subjectivity. In doubtful cases, when establishing the data used in this article, the author consulted a professional organist for his opinion on the definition of the subject.

Compass, x_2. The compass of the subject is measured in semitones in this article. Thus, for instance, for a perfect fifth $x_2 = 7$ and an octave yields $x_2 = 12$. For vocal fugues, the compass is often within an octave (due to limitations of the human voice). However, also for instrumental fugues, too large a compass is hard to manage, if one wishes to avoid crossings of voices.

Number of pitch classes, x_3. Consider the conventional 12-note scale, and the notion of octave equivalence. Hence this variable has an upper bound: $x_3 \leq 12$. Among Bach's subjects, we find a maximum value of $x_3 = 11$ (for the fugue in E minor, BWV 548). For a theme of diatonic character, a value of x_3 below 7 is likely.

Variables x_4, x_5, x_6. Regarding x_4, perhaps we expect no distinguishing effect due to composer; the interval between the first two notes would probably not depend on the era in music history. However, when studied in relationship to other interval related variables, it might be of interest. Obviously, $x_4 \geq 0$, where $x_4 = 0$ means a note repetition. The maximum initial interval observed overall is $x_4 = 12$ (an octave). Turning to x_5, note that unisons are included and complementary intervals are distinguished, but intervals are not distinguished by direction. We find the minimum observed $x_5 = 1$ in a subject by Josef Rheinberger, *Fuga cromatica*, which is simply a chromatic scale, hence only one type of interval occurs (a semitone). See Figure 2 for this subject.

Figure 3. Subject from Lemare's fugue in D minor, op. 98.

Figure 4. Bach: subject from D major fugue (BWV 532).

For x_6, the maximum interval between successive notes, obviously by definition $x_6 \leq x_2$. Again, the minimum value $x_6 = 1$ is found in the Rheinberger subject referred to above, while the maximum is encountered in a subject by Edwin H. Lemare, $x_6 = 17$ (and $x_2 = 19$); see Figure 3.

Some remarks. The answer of the subject could be of so-called tonal or real type. I have used the form of the subject as presented initially. Moreover, I have not studied so-called countersubjects, i.e. a counterpoint which accompanies the subject or answer systematically.

Among Bach's fugues, musicologists identify certain types of writing. Stauffer (1986) mentions e.g. the types *Spielfuge*, the dance fugue, the allabreve fugue, and the art fugue. The *Spielfugue* could imply longer subjects and often sequential passages. A famous example is the D major fugue (BWV 532) whose structure is built upon sequential passages. Despite the considerable length of such a subject, few unique notes are employed (see Figure 4).

The length of a subject is in this paper measured in number of written (or in performance, played) notes. Length could have been measured in number of bars, but type of fugue (cf. the preceding paragraph) or tempo could have an influence. Optionally, an additional variable with the metre of the piece could have been introduced, but the author strived for the simplest possible solution, working with integers related directly to the subject.

3. Introductory data analysis

3.1. *Data collection*

Data were collected based on samples from the author's library of sheet music for organ, and from digital scores available online at IMSLP Petrucci Music Library. The intention was to cover the main fugues from the organ repertoire from Bach and onwards, but non-mainstream works have been included as well, including e.g. some probably lesser known Swedish composers. Fugues with a conventional structure (subject presented alone (*dux*), then the response (*comes*)) were chosen, for the sake of identification of themes. More modern twentieth-century works were not analysed (e.g. *12 Orgelfugen durch alle Tonarten* by Johann Nepomuk David, written 1967–1968). A list of the works by "other composers" in the data set is found in Appendix 3.

Identification of subjects was made by the author, in some cases with assistance from a professional organist. In all, 238 fugue subjects were collected and features compiled (Bach: 47; Reger: 45; Others: 146). Ornaments, grace notes, etc., were not counted as being notes of the subject. In the case of double fugues (with two subjects), both of these were collected and regarded as two separate subjects. In the case of composer Charles-Marie Widor, he revised his music extensively.

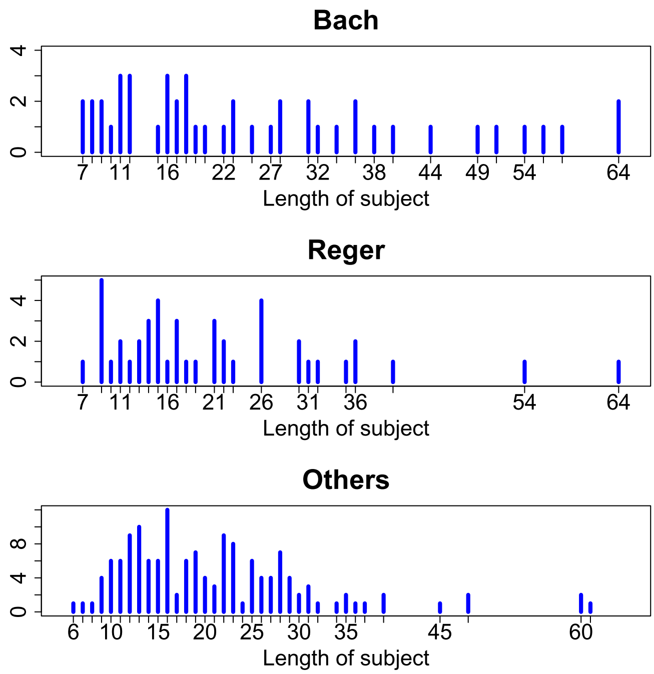

Figure 5. Histogram of X_1, length of subject.

A fugue in E minor was present in the first edition of the third organ symphony, but omitted in later revision work. I have chosen to include that fugue in the data set, as it is still an example of a fugue written by this composer (albeit not eventually placed in a symphony).

3.2. *Visualisations*

We examine closer the considered quantities $x_1, x_2, \ldots, x_6$, now regarded as observations of related random variables $X_1, X_2, \ldots, X_6$. These are all integer-valued variables, and for X_1, X_2, X_3 we first investigate distributional aspects in the form of histograms, shown in Figures 5–7. The empirical distributions of the remaining variables are briefly discussed. Thereafter, tests for differences in distributions are performed.

3.2.1. *Discussion of variables* X_1, X_2, X_3

Concerning X_1, length of subject, we note from Figure 5 a considerable variability, a result of the creative process of creating a fugue theme. Distributions are right-skewed, for obvious reasons. Among Bach's works, we find the longest subjects (with a length of 64 tones) as the D major fugue BWV 532, which is of so-called *Spielfuge* type and the G major fugue BWV 577 (dance fugue, "gigue"). The longest subject among other composers is the second fugue on BACH by Robert Schumann. As pointed out by Stinson (2006), the construction of this theme has similarities in the overall form with Bach's fugue subject of BWV 575 (Fugue in C minor, also quite long, 54 notes). At a closer level, there are also melodic fragments similar to Bach's fugue BWV 565. This is an example where later-time composers give a nod at Bach and his oeuvres.

For the variable X_2, range, we note in Figure 6 somewhat more symmetric shapes of the distributions with a typical mode around the octave (12 semitones). The greatest range of Bach's

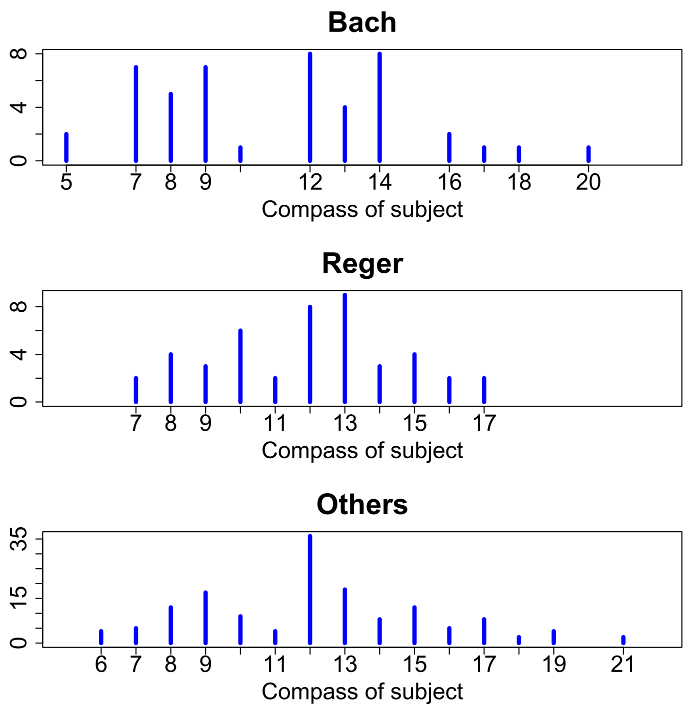

Figure 6. Histogram of X_2, range of subject.

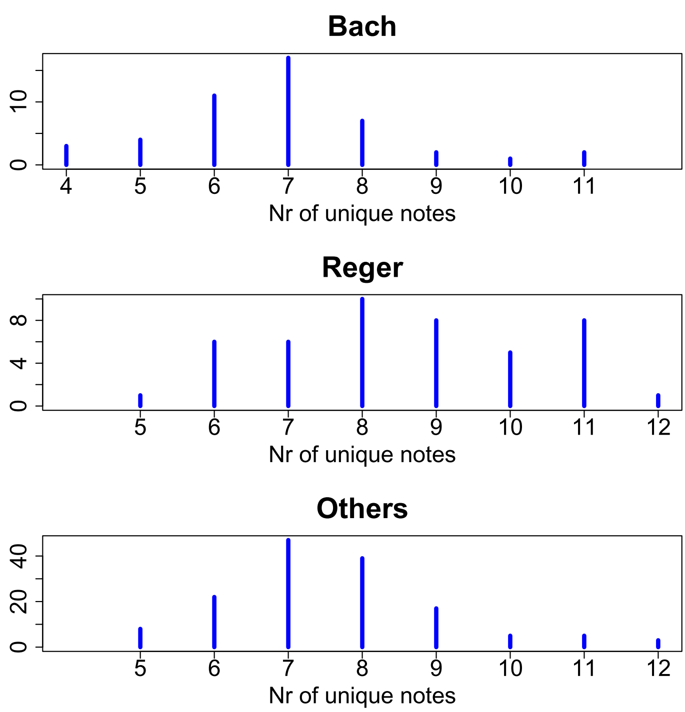

Figure 7. Histogram of X_3, number of unique pitch classes of subject.

fugues is in BWV 577 – a fugue which also has a considerable length. (The statistical relationships between the variables, a key issue in this paper, are examined in Sections 4 and 5). Among the group of other composers, the largest range is found in the B major fugue, opus 7:1 by Marcel Dupré (1886–1971) and an A minor fugue by Achille Philip (1878–1959).

Turning to X_3, the number of unique pitch classes, we have as a consequence of the chromatic scale an upper bound 12. Most fugues in the data set, regardless of composer category, has typically 7 or 8 notes, as seen from Figure 7. A composer may intentionally aim for writing a fugue using all 12 notes. Both Josef G. Rheinberger (1839–1901) and Charles V. Stanford (1852–1924) labelled fugues *Fuga cromatica*. From Figure 7, it seems that outcomes of Reger are shifted against higher values, but again the relations to X_1 and X_2 need to be taken into account for deeper conclusions (Sections 4 and 5).

3.2.2. *Discussion of variables* X_4, X_5, X_6

For clearer presentation, histograms of these variables are found in the Appendix as Figures A1–A3. For all categories, regarding the initial interval as described by variable X_4, one notes that most outcomes are $0 \leq x_4 \leq 7$, or $x_4 = 12$ (an octave). Curiously, for Reger's subjects, $0 \leq x_4 \leq 5$ or $x_4 = 12$; in other words, no subject has an opening fifth. Turning to X_5, the number of unique intervals between notes, the mode value is $x_5 = 5$ for all categories; however, interestingly Bach has as many as four subjects where $x_5 = 11$: BWV 543, BWV 548, BWV 564, BWV 575. In the total collection of subjects, we find further only Schumann's second fugue on BACH op 60:2 with $x_5 = 11$. Finally, for X_6, the maximum observed value for Bach and Reger is $x_6 = 12$. Interestingly, the *minimum* observed maximum interval between successive notes with Reger is as high as $x_6 = 5$.

3.3. *Statistical tests for shifts in distributions*

We now test for possible differences in distribution between the three categories for each of the variables. Since the distribution families are not known and difficult to assess, a non-parametric approach is taken and we here use Kruskal–Wallis test (see e.g. Conover 1999). This is a rank-sum test of the null hypothesis that the location parameters of the distribution are the same, the alternative that they differ for at least one sample.

Suppose we have g groups and the total number of observations $N = n_1 + \cdots + n_g$. Furthermore, let r_{ij} be the rank of observation j in group i and $r_{i\bullet}$ be the sum of the ranks in group i. In our case, we have tied observations, i.e. observations having the same value. Then, the average rank is assigned to each of the tied observations.

The test statistic is given by

$$K = \frac{1}{S^2}\left(\sum_{i=1}^{g} \frac{r_{i\bullet}^2}{n_i} - \frac{N(N+1)^2}{4}\right),$$

where

$$S^2 = \frac{1}{N-1}\left(\sum_{i=1}^{g}\sum_{j=1}^{n_i} r_{ij}^2 - \frac{N(N+1)^2}{4}\right).$$

Here, S^2 is the variance of the ranks (in the case of no ties, $S^2 = N(N+1)/12$). For decision making, the outcome of K is compared to a quantile from the $\chi^2(g-1)$ distribution. Often p values are reported when the results of the tests are presented; a p value is the probability of obtaining an effect at least as extreme as the one observed in the sample. A p value less than some specified value (usually 0.05, i.e. the significance level of 0.05) leads to rejection of the null hypothesis, and the conclusion of a present effect.

Calculations were made, using the routine `kruskal.test` in the statistical software R (R Core Team 2018), which handles ties. We find p values for the analysis of differences between categories in the cases of X_1, X_2 and X_3 as 0.44, 0.19 and 8.5×10^{-6}. In other words, Kruskal–Wallis test reports no differences between the composer categories regarding X_1 or X_2, while differences exist for X_3 (not unreasonable, cf. Figure 7). To find between which groups there are differences, conventionally a suitable test for multiple comparisons (also called post-hoc test) is employed. We here use Dunn's test (Dunn 1961) as implemented in the function `dunn.test` from the R package with the same name, and find significant differences between all three categories. For variables X_4, X_5, X_6, for each variable no significant differences were found between categories, using the same methodology as above.

Note that in particular the variables X_4 and X_6 (cf. Figures A1 and A3) take few values, in other words, their distributions seem to be sparse. The use of Kruskal–Wallis test might not be warranted.

4. Multivariate modelling

In this section, we investigate statistical relationships between the three variables considered by approaches from the field of multivariate statistics or, alternatively phrased, statistical learning. Within the latter paradigm, we here face a situation of so-called unsupervised learning, since we have, at least at this stage in the investigation, no specific input or output variables. More on the techniques can be found e.g. in James et al. (2013).

4.1. *Principal components*

By an analysis of principal components (PCA), a low-dimensional representation of a data set is found, which captures as much information of the variation as possible. Overall aims are usually data reduction (in the case of many variables) and interpretation. In our case, the latter is in focus since we only consider six variables. Mathematically speaking, linear combinations of the variables are studied. In our case, the first principal component Z_1 of the variables $X_1, X_2, \dots, X_6$ is the normalised linear combination

$$Z_1 = a_{11}X_1 + a_{21}X_2 + \cdots + a_{61}X_6$$

which has the largest variance; normalised, in the sense of

$$\sum_{j=1}^{6} a_{j1}^2 = 1.$$

For observations, the first sample principal component is obtained, notated by Z_1. The following associated optimisation problem is solved, with x_{ij} being the sample values, or observations,

$$\max_{a_{11},\dots,a_{61}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{6} a_{j1} x_{ij} \right)^2 \right\} \quad \text{subject to} \quad \sum_{j=1}^{6} a_{j1}^2 = 1.$$

After the first principal component Z_1 has been deduced, the second principal component, Z_2, is found. This is the linear combination which has the maximal variance out of all linear combinations uncorrelated with Z_1.

In practice, software readily performs the optimisation in the PCA. There is also a connection to an eigenvalue-eigenvector problem for the sample covariance (or correlation) matrix of the involved variables, see e.g. Johnson and Wichern (2007).

4.2. *Results for the data set*

For our data, we find using the software R and the routine `prcomp` the following three first principal components:

	x_1	x_2	x_3	x_4	x_5	x_6
z_1	0.44	0.47	0.31	0.19	0.49	0.46
z_2	−0.19	−0.040	−0.49	0.82	−0.0067	0.22
z_3	0.33	−0.055	−0.78	−0.41	0.32	0.08

Typically cumulative percentage of the total variance is reported. For our data, the first principal component accounts for 51%, the first two principal components 68% and the first three 80%. Moreover, usually an attempt is made of interpreting the coefficients, or loadings, as these are occasionally called. Here, the first essentially is a weighted linear combination of the variables, with positive weights. Less weight is put on x_4, initial interval in semitones. The second contrasts the x_1 and x_3 (length and number of pitch classes, in a sense overall measures of the subject) against x_4 and x_6 (interval features, inner construction of subject).

A visualisation is often made by plotting the observations in a plane spanned by the two principal components. In Figure 8, categories do not separate completely; however, we may note a tendency for Bach observations to fall into the space of lower values of z_1, and Reger observations seem to receive lower values on z_2 than most of Bach's cases. Figure 9 gives the related numbers of items, and makes it possible to discuss selected works, which is done next.

When relating the observation numbers in Figure 9 to the musical contents of the actual works, we may roughly interpret tendencies in various regions of the $z_1 - z_2$ plane as follows.

Upper-left Shorter themes, mostly diatonic nature.

Upper-right Longer themes, mostly diatonic nature.

Lower-left Shorter themes, mostly chromatic nature. No Bach works; composers like Liszt (the BACH fugue), Brahms (A flat minor fugue), Reger (fugue from 2nd organ sonata).

Lower-right Longer themes, often chromatic nature. Reubke (fugue from organ sonata), Reger (2nd fugue subject, op. 135b), Willan (second fugue subject, Prelude and Fugue in C minor). Dawes (2005) has pointed out common features of Willan's early larger organ works with Reger.

Note that the descriptions of subjects as being of (mostly) diatonic or chromatic nature were made by the author reading and analysing the score, not by quantitative methodology.

4.2.1. *Discussion of selected works*

Let us, based on Figures 8 and 9, consider some observations that seem to deviate somewhat from their respective regions.

Bach: obs. 9, 30, 45. These are compositions BWV 543, BWV 575 and BWV 577 and are found in the region of high z_1 and low z_2. They have in common considerable lengths.

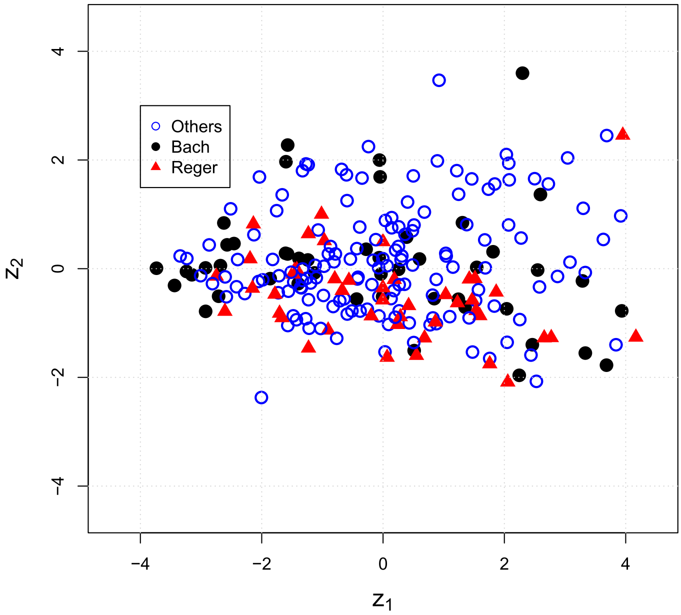

Figure 8. Observations plotted on the two first principal components (categories shown).

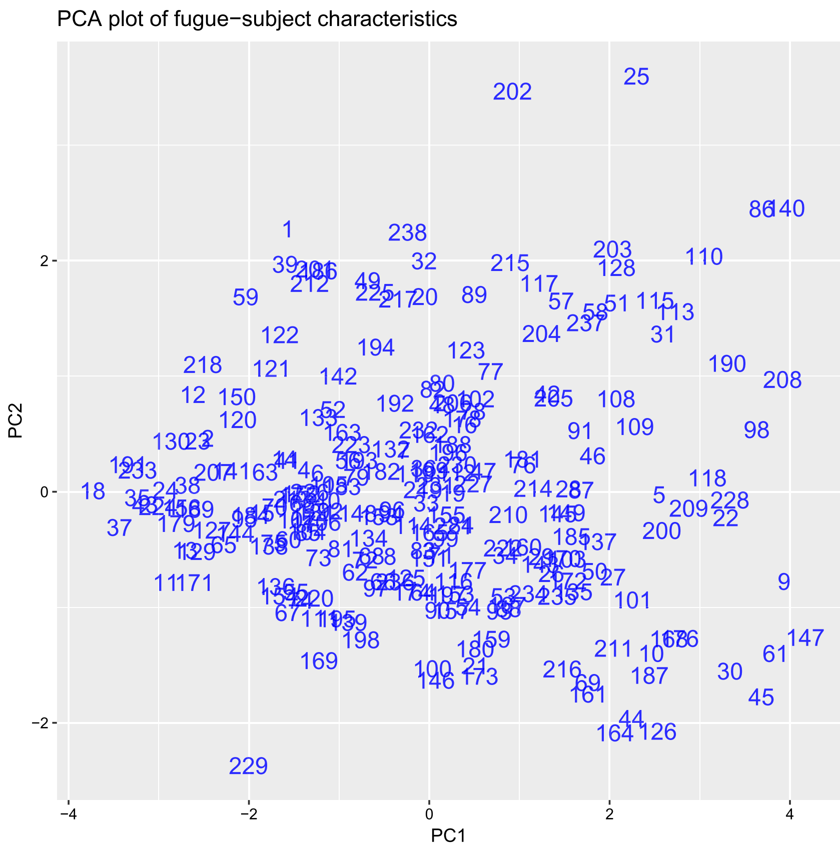

Figure 9. Observations plotted on the two first principal components (indexed works, for reference).

Figure 10. Bach: C major fugue, BWV 531.

Figure 11. Reger: C major fugue, op. 7.

Figure 12. Reger subjects: op. 59:6, op. 7, op. 69:10.

Figure 13. Stanford: C minor fugue op. 193:2.

Bach: obs. nr 25. This is BWV 531, with the highest value of z_2 of all items; see Figure 10 for the subject. It stands out somewhat from the other Bach works, and on stylistic grounds, this might be reasonable. In fact, for this work by Bach from his early production, quoting Williams (2003):

> Clearly, the work is an early and imaginative response to the music of established masters, with marked similarities in figuration, texture, harmony and use of the organ, all of these implying a common genre.

Reger: obs. 140. This is a fugue in C major, op. 7, high z_1, high z_2, with a long theme ($x_1 = 64$) but relatively few unique notes ($x_3 = 8$). See Figure 11 for the subject. This could be interpreted as a subject of type *Spielfuge* that its length does not show relatively many unique notes – a mostly diatonic theme.

Reger: obs. 133, 142, 163. These are located middle low z_1, middle high z_2, and are Reger fugues with opus 59:6, opus 7 (D minor, second subject) and opus 69:10. See Figure 12 for the subjects. These are quite early works in Reger's production.

Others: obs. 202, 229. For high z_2, obs. 202 is the C minor fugue op. 193:2 by C.V. Stanford, see Figure 13. For low z_1 and low z_2, obs. 229 is the subject from *Fuga cromatica* by Rheinberger discussed above (see Figure 2).

4.2.2. *A remark on scaling*

Finally, a remark on scaling of data. A scaling to unit variance is often strongly recommended, see e.g. James et al. (2013, Section 10.2.3). If not done, the magnitudes of the variables (or units chosen) may influence the result. The results given above were obtained after scaling. If not performing scaling, the first principal component would be dominated by X_1 due to its in general larger numbers (The loading on variable X_1 is then as high as 0.974).

5. Regression models

We now consider regression models for the analysis. Preliminary investigations show that for all three composer categories, the correlations between X_1, X_2 and X_3 are positive. In regression models, the response variable is of crucial importance and we consider two situations, where X_2 and X_3, respectively, act as response variables. Of primary interest is then to see whether there is a difference between the three categories of composers.

5.1. *Range of fugue subject*

We here consider a regression model with X_2 as a response. As this is a count variable, we might choose *Poisson regression* as a first option. In the general case, a model of the following type is then fitted (with, as usual, lower-level letters denoting sample values of variables):

$$\mu_i = \exp(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}), \quad i = 1, \ldots, n,$$

where μ_i is the mean response for the ith case, and there are p covariates. In our application, we might consider the covariate X_1 (length) entering per se, or in the form of a so-called offset term. Such modelling is often encountered in statistical risk analysis, then in the context of exposure time (Rychlik and Rydén 2006). Furthermore, the categories enter as so-called dummy variables. The three categories imply, using treatment coding, that two dummy variables need to be introduced. Goodness of fit is often checked by examining closer the residual deviance of the fitted model, see e.g. Madsen and Thyregod (2011). Here, the residual deviance is $D = 303.48$ in a model with 231 degrees of freedom (this might indicate a problem with so-called overdispersion). Let $Q \sim \chi^2(231)$; then $\mathsf{P}(Q \geq 303.48) = 9.6 \times 10^{-4}$. Hence, the model does not fit adequately, and another option must be taken.

Often a *negative-binomial* response is considered in regression models for counts, when the simpler Poisson assumption fails (Hilbe 2011). With X_1 as an offset, we then obtain, using the routine `glm.nb` in the R package MASS, a model with deviance $D = 229.86$ on 231 degrees of freedom, which yields a good fit: if again $Q \sim \chi^2(231)$, $\mathsf{P}(Q > 229.86) = 0.51$. A summary of the fitted model is given in the Appendix, as well as a plot of the so-called deviance residuals in Figure A4, for the purpose of model diagnostics. The R environment requires that a baseline reference is given for the dummy variables. With "Bach" category as the reference level for the dummy variables, we find significant differences to "Reger" as well as "Others" (p values 4.4×10^{-5}, 9.5×10^{-6}, respectively). Redefining the baseline level and performing the estimation, one finds in addition no significant difference between "Others" and "Reger."

From the summary, we note further that X_4 (initial interval), is clearly non-significant (p value 0.9640); recall the reflection upon this in Section 2. Variables X_3, X_5 and X_6, on the other hand, are significant in the model.

5.2. *Number of unique pitch classes*

We now consider X_3 as the response variable and investigate its dependence on other variables. By definition, $X_3 \leq 12$. However, in practice X_3 cannot attain the value zero (no music), and a fugue subject containing only one note would be peculiar; entirely rhythmic in nature and so far not encountered in music to the author's knowledge. The minimum observed value in the dataset is $x_3 = 4$ (for BWV 553).

Modelling of the response is not straightforward, and we discuss various approaches in a subsection below. At present, we interpret X_3 as a multi-category response with 12 categories $(1, 2, \ldots, 12)$, and use an ordinal-response regression model (Agresti 2013, Chapter 8.2, Bilder and Loughin 2015, Chapter 3.4) with cumulative logits, as follows.

We have $J = 12$ categories, and the cumulative probability for category j of X_3 is

$$\mathsf{P}(X_3 \leq j) = \pi_1 + \cdots + \pi_j, \quad j = 1, \ldots, J.$$

(Note that $\mathsf{P}(X_3 \leq J) = 1$.) In the regression model, the cumulative logits are introduced:

$$\text{logit}(\mathsf{P}(X_3 \leq j)) = \log\left(\frac{\mathsf{P}(X_3 \leq j)}{1 - \mathsf{P}(X_3 \leq j)}\right) = \log\left(\frac{\pi_1 + \cdots + \pi_j}{\pi_{j+1} + \cdots + \pi_J}\right).$$

The commonly used proportional odds model assumes that the logit of these cumulative probabilities changes linearly as the explanatory variables change. Furthermore, it is assumed that the slope is the same regardless of the category j. The model is, for p explanatory variables,

$$\text{logit}(\mathsf{P}(X_3 \leq j)) = \beta_{j0} + \beta_1 x_1 + \cdots + \beta_p x_p, \quad j = 1, \ldots, J - 1.$$

By the routine `polr` in the R package `MASS`, a model can be fitted with covariates X_1, X_2, X_4, X_5, X_6 and dummy variables for category of composer. The composer category turns out to be significant (in the resulting ANOVA table, $p = 2.0 \times 10^{-7}$), and closer examination shows differences between all composers. This is in line with the results found for X_3 by the non-parametric tests in Section 3.3.

5.2.1. *Other regression approaches*

One might consider a Poisson distribution for X_3, although there is an upper limit of the variable, and proceed with Poisson regression. Alternatively, a truncated Poisson distribution might be investigated, with $1 \leq X_3 \leq 12$. However, this is not straightforward to use in R (the case with non-zero truncation, $X_3 > 0$, is on the other hand common and implemented). Moreover, one could consider each work a binomial distribution, $X_3 \sim \text{Bin}(12, p)$, where the related random experiment is interpreted as a composer choosing among 12 notes, each one chosen independently with probability p (The independence assumption is highly questionable in tonal music). This leads to a model of logistic regression. Possibly a truncated model could be introduced. To conclude, the author tried conventional Poisson regression and logistic regression for the data set, and for these models, the composer category turned out to be highly significant, cf. the proportional odds model above.

6. Discussion

In this paper, we characterized fugue subjects by six integers and examined statistical relationships between these for a large selection of subjects, and taking into account the composer

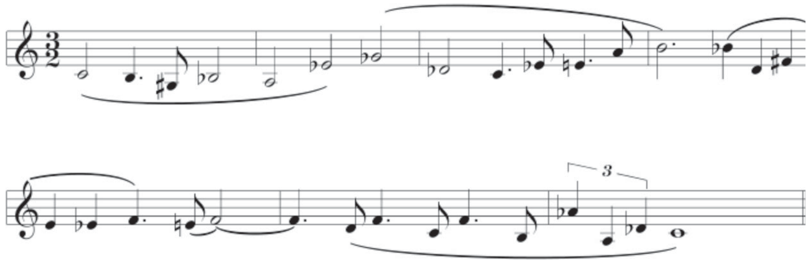

Figure 14. Valen: Fugue subject (op. 33).

background of the work. Obviously, this is a simplification: to fully describe a fugue subject would need descriptions in terms of harmonic implications, rhythmic features, etc. However, the quantification implies that several statistical methodologies can be employed. The visualisation of subjects is possible by the PCA and might be used as one possible means to judge doubtful compositions or to date works to certain periods; recall for instance the discussion of subject BWV 531 in Section 4.2.

A further reflection could be made on the coding of intervals to numbers, essential for the analysis made in the paper. In the paper, an interval between two consecutive notes was measured in semitones, resulting in an integer. However, this might be a simplification of the musical and harmonic meaning. Consider for instance the interval of a major sixth, resulting in the integer 9. In a harmonic context, however, this number could as well correspond to a diminished seventh which, not the least in Bach's music, often adds a dramatic flavour. An example is found in the subject in Figure 1, where the interval B3 to A4 flat is found. An alternative coding could take into account the actual function of an interval, not merely the number of semitones.

Consider a subject by Norwegian composer Fartein Valen, shown in Figure 14. This is created within the framework of free tonality, yet constructed in a traditional way with entries *dux*, *comes*, etc. The length is $x_1 = 20$ and there are thus 19 intervals between successive notes. The following intervals are found: 1, 2, 3, 4, 5, 6, 8, 9, 11. The most common interval is 1 (a minor second), 9 out of the 19 intervals. In this musical style, the coding used in the present paper works fine.

Regarding range X_2, we found from the final regression model in Section 5.1 that Bach differs from the other composers. From the histograms in Figure 6, we may view the particular spreading of values from Bach's observations, at least compared to the outcomes of Reger (which is about as large) and also "gaps" for some integers. Perhaps the result is a simple consequence of the limited number of Bach works, compared to the bulk of subjects from the category "Others." The more works considered, the more integers get covered in the empirical distribution. Note, however, that the tests by Kruskal–Wallis in Section 3.3 on x_2 data alone resulted in no statistically significant differences.

Regarding the number of pitch classes X_3, the proportional odds model in Section 5.2 confirmed the findings from the simpler tests in Section 3.3: differences between all categories.

The quantification of diatonic versus chromatic subjects is not trivial. The discussion of Figure 8 was based on a direct interpretation of the score by the author. One could surmise that fewer pitch-classes usually means more diatonic; however, it is possible that a composer could limit the number of unique pitch-classes by writing a completely chromatic theme in a limited range, for instance, or one based on, say, a single diminished seventh chord.

For further research, one could think of a measure (or index), describing the extent of chromaticism in a fugue subject (cf. Perttu 2007). Such a measure should take into respect how "crammed" notes are within the compass (Quinn 2006, 2007). The number, or possibly ratio, of non-diatonic notes in the key, could be another indicator. Still another aspect for comparison could be the very distribution of notes, leading to analysis of circular data (see Beran 2004, Chapter 7) over the conventional 12-note scale. Such an analysis is, of course, not restricted to the analysis of particularly fugue subjects, see Beran (2004) for examples.

In this paper, we studied fugue subjects in music history after Bach. An option is to analyse in another study the predecessors of Bach, e.g. organ fugue-subjects by Buxtehude.

Acknowledgments

The author is grateful to the co-editor-in-chief, Jason Yust, for valuable comments and remarks. Moreover, thanks to Pontus Leitz, organist at the parish of Gamla Uppsala, for assistance with definitions of fugue subjects; Sigvard Selinus, organist at St. John's Church, Stockholm, for discussion on interpretations of intervals; and to Rolf Larsson, Dept. of Mathematics, Uppsala University, for comments on the manuscript.

Disclosure statement

No potential conflict of interest was reported by the author.

References

Agresti, Alan. 2013. *Categorical Data Analysis*. 3rd ed. Hoboken, New Jersey: Wiley.
Backer, Eric, and Peter van Kranenburg. 2005. "On Musical Stylometry – A Pattern Recognition Approach." *Pattern Recognition Letters* 26 (3): 299–309.
Beran, Jan. 2004. *Statistics in Musicology*. Boca Raton: Chapman & Hall/CRC Press.
Beran, Jan, and Guerino Mazzola. 1999. "Analyzing Musical Structure and Performance – A Statistical Approach." *Statistical Science* 14 (1): 47–79.
Bilder, Christopher R., and Thomas M. Loughin. 2015. *Analysis of Categorical Data with R*. Boca Raton: Chapman & Hall/CRC Press.
Conover, William J. 1999. *Practical Nonparametric Statistics*. 3rd ed. Hoboken, New Jersey: Wiley.
Dawes, Chris. 2005. "Theatricality in Healey Willan's Introduction, Passacaglia and Fugue for Organ (1916)." *McMaster Music Analysis Colloquium* 4 (1): 52–73.
Dunn, Olive J. 1961. "Multiple Comparisons Among Means." *Journal of the American Statistical Association* 56 (293): 52–64.
Gédalge, André. 1901. *Traité de la fugue*. Paris: Enoch & Cie.
Hilbe, Joseph M. 2011. *Negative Binomial Regression*. 2nd ed. Cambridge: Cambridge University Press.
James, Gareth, Daniela Witten, Trevor Hastie, and Robert Tibshirani. 2013. *An Introduction to Statistical Learning with Applications in R*. New York: Springer-Verlag.
Johnson, Richard A., and Dean W. Wichern. 2007. *Applied Multivariate Statistical Analysis*. 6th ed. New Jersey: Pearson Education International.
Madsen, Henrik, and Poul Thyregod. 2011. *Introduction to General and Generalized Linear Models*. Boca Raton: Chapman & Hall/CRC Press.
Mann, Alfred. 1987. *The Study of Fugue*. New York: Dover Publications.
Perttu, Daniel. 2007. "A Quantitative Study of Chromaticism: Changes Observed in Historical Eras and Individual Composers." *Empirical Musicology Review* 2 (2): 47–54.
Prout, Ebenezer. 1891. *Fugue*. 4th ed. London: Augener & Co.
Quinn, Ian. 2006. "General Equal-Tempered Harmony (Introduction and Part I)." *Perspectives of New Music* 44 (2): 114–158.
Quinn, Ian. 2007. "General Equal-Tempered Harmony (Parts 2 and 3)." *Perspectives of New Music* 45 (1): 4–63.
R Core Team. 2018. *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.
Rychlik, Igor, and Jesper Rydén. 2006. *Probability and Risk Analysis. An Introduction for Engineers*. Berlin: Springer-Verlag.
Rydén, Jesper. 2006. "Statistical Analysis Of Golden-Ratio Forms in Piano Sonatas by Mozart and Haydn." *The Mathematical Scientist* 32 (1): 1–5.
Stauffer, George. 1986. "Fugue Types in Bach's Free Organ Works." In *J.S. Bach as Organist*, edited by George Stauffer and Ernest May. Bloomington: Indiana University Press.

Stinson, Russell. 2006. *The Reception of Bach's Organ Works from Mendelssohn to Brahms*. New York: Oxford University Press.
Tovey, Donald F. 1924. *Forty-Eight Preludes and Fugues by JS Bach* (editor). London: Associated Board of the Royal Schools of Music.
Williams, Peter. 2003. *The Organ Music of J.S. Bach*. 2nd ed. Cambridge: Cambridge University Press.

Appendix 1. Histograms for variables X_4, X_5, and X_6

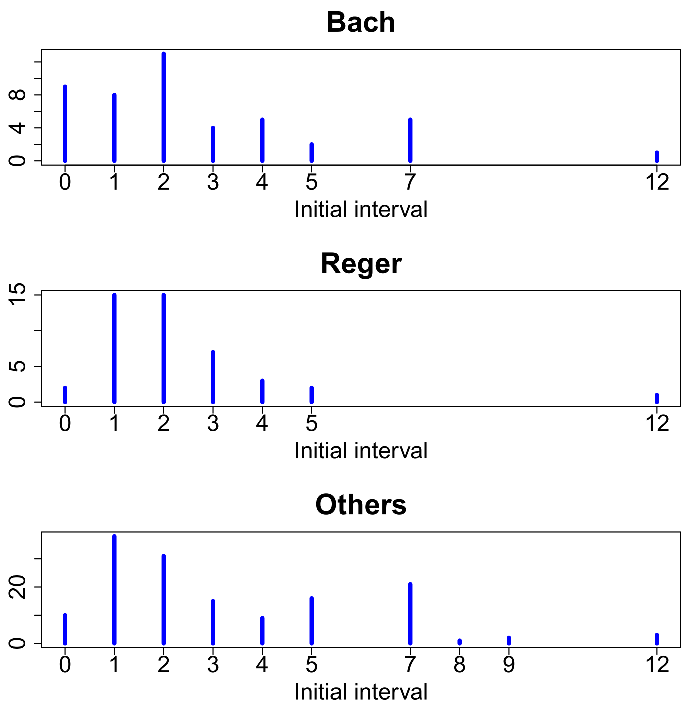

Figure A1. Histogram of X_4, initial interval.

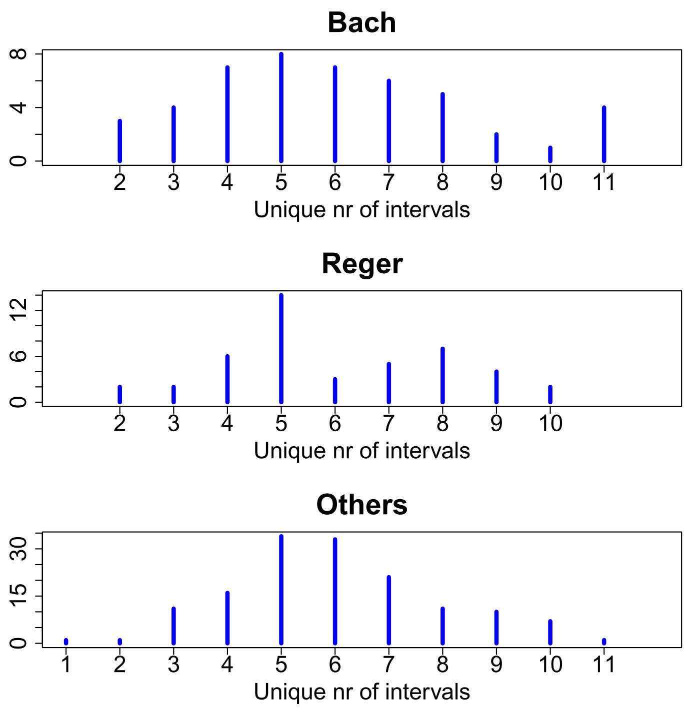

Figure A2. Histogram of X_5, unique number of intervals.

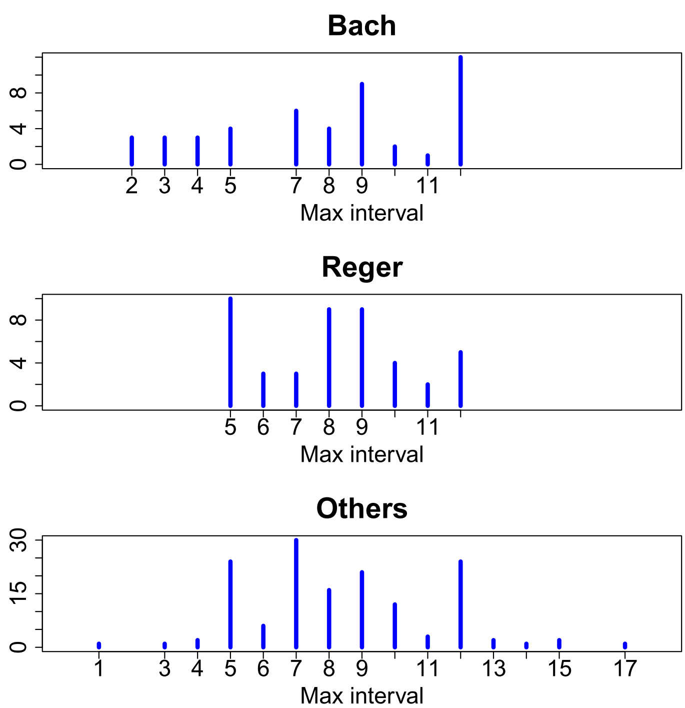

Figure A3. Histogram of X_6, maximum interval between successive notes.

Appendix 2. Fitted regression model

Histograms for variables X_4, X_5, and X_6 Model with range X_2 as response. Regression summary from R after fitting a negative binomial model.

```
 glm.nb(formula = V3 ~ V4 + V5 + V6 + V8 + V13 + offset(log(V2)),
 data = bada, init.theta = 39.31385261, link = log)
 Coefficients:
Estimate Std. Error z value Pr(&gt;|z|)
 (Intercept) -0.121608 0.122096 -0.996 0.3192
 V4 -0.033485 0.015343 -2.182 0.0291 *
 V5 -0.000401 0.008879 -0.045 0.9640
 V6 -0.127971 0.014488 -8.833 < 2e-16 ***
 V8 0.047483 0.010743 4.420 9.88e-06 ***
 V132 0.306058 0.074899 4.086 4.38e-05 ***
 V130 0.259553 0.058608 4.429 9.48e-06 ***
 ---
 Signif. codes: 0 '***' 0.001 '**' 0.01 '*'
 0.05 '.' 0.1 ' ' 1
 (Dispersion parameter for Negative Binomial(39.3139) family taken to be 1)
 Null deviance: 356.83 on 237 degrees of freedom
 Residual deviance: 229.86 on 231 degrees of freedom
 AIC: 1329.2
```

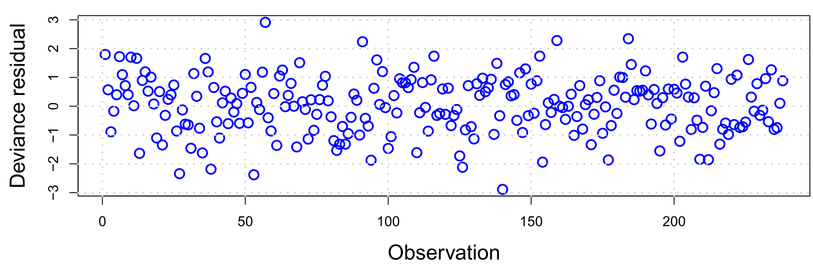

Figure A4. Deviance residuals after fitting a negative binomial model.

Appendix 3. Fugue themes: other composers

A–C	
G.W. Andrews	Fugue in A minor
A. Barié	Fugue from Organ symphony (op. 5)
J.G. Bastiaans	Double fugue, G minor
A. Becker	Prelude and fugue (op. 9)
H. Bellermann	Fugue on BACH (op. 8)
E. Bernard	Fantaisie et fugue (op. 24)
W.T. Best	Fantasie and fugue, A minor
L. Boëllmann	Fugue (op. 16)
J. Brahms	Prelude and fugue, A minor
J. Brahms	Prelude and fugue, G minor
J. Brahms	Fugue, A flat minor
A. Bruckner	Fugue, D minor
F. Capocci	Fugue, G major
G. Catoire	Prelude and fugue (op. 25)
G. Couture	Fugue, D minor
C. Czerny	Prelude and fugue, A minor (op. 603:3)
C. Czerny	Prelude and fugue, D minor (op. 603:6)
C. Czerny	Prelude and fugue, A minor (op. 607)
D–G	
J.N. David	Prelude and fugue, A minor
J.N. David	Prelude and fugue, G major
R. Diggle	A joyous fugue
T. Dubois	Prelude and fugue (12 pièces nouvelles)
M. Dupré	Three preludes and fugues (op. 7)
M. Dupré	Three preludes and fugues (op. 36)
M. Dupré	Four modal fugues (op. 63)
M. Duruflé	Prelude and fugue on the name Alain
H. Eslava	Ofertorio (Fugue), D minor
J.A. van Eyken	Toccata and fugue on BACH
W. Faulkes	Concert prelude and fugue
W. Faulkes	Prelude and fugue, G minor
A. Fleury	Prelude and fugue, F minor
C. Franck	Prélude, fugue et variation
H. Fryklöf	Fugue, E minor
A. Guilmant	Fugues from Sonatas 3, 5 and 6
A. Guilmant	Fugue, A flat major (op. 40:1)
A. Guilmant	Prelude and fugue, E minor (op. 58:1)
A. Guilmant	Fugue, F minor (op. 90:7)
H–L	
A.F. Hesse	Two fugues (op. 39)
A. Honegger	Fugue et choral
G. Hägg	Fugue, G minor
E. Köhler	Prelude and fuge, F major
E. Köhler	Prelude and fuge, A major
E. Köhler	Prelude and fuge, D major
E.H. Lemare	Toccata and fugue, D minor (op. 98)
E.H. Lemare	Scherzo fugue (op. 102)
J.N. Lemmens	Fugue, B minor
J.N. Lemmens	Fugue, C minor
J.N. Lemmens	Fugue, D major ("Fanfare")
J.N. Lemmens	Fugue, F minor
O. Lindberg	Prelude and fugue, A minor
F. Liszt	Prelude and Fugue on BACH (S260)
F. Liszt	Fantasy and fugue on "Ad nos, ad salutarem undam" (S259)
G. Litaize	Double fugue

(*Continued*).

Continued.

M–R	
F. Mendelssohn	Three preludes and fugues (op. 37)
F. Mendelssohn	Fugue, C major (Sonata 2)
F. Mendelssohn	Fugue, D minor (Sonata 6)
G. Merkel	Fugue on BACH, op. 40
H.W. Nicholl	Preludes and fugues (op. 33)
H.W. Nicholl	Preludes and fugues (op. 35)
J. Nyvall	Variations and fugue on “Vår blick mot helga berget går”
D. Olson	Prelude and fugue, F sharp minor
O. Olsson	Prelude and fugue, C sharp minor (op. 39)
O. Olsson	Prelude and fugue, F sharp minor (op. 52)
O. Olsson	Prelude and fugue, D sharp minor (op. 56)
O. Olsson	Fantasy and fugue, “Vi lofve dig, o store Gud” (op. 29)
T.I. Pachaly	Fugue on BACH
H. Parker	Fugue, C minor (op. 36:3)
R.L. Pearsall	Introduction and fugue, D minor
A. Philip	Toccata and fugue, A minor
C. Quef	Fugue, E minor
J. Reubke	Sonata on the 94th Psalm
J.G. Rheinberger	Fugues from Sonatas 1–4, 6–7, 9–13, 16, 17
J.G. Rheinberger	Fughetta on BACH (op. 123:3)
J.G. Ropartz	Fugue, E minor
S–Å	
C Saint-Saëns	Preludes and fugues (op. 99)
C Saint-Saëns	Preludes and fugues (op. 109)
C. Schumann	Fugue from Sonata B-flat major (op. 16:4)
R. Schumann	6 fugues on BACH (op. 60)
E. Sjögren	Prelude and fugue, G minor
E. Sjögren	Prelude and fugue, A minor
C.V. Stanford	Prelude and fugue, E minor
C.V. Stanford	Three preludes and fugues (op. 193)
G. Thyrestam	Prelude and fugue, C sharp minor
G. Thyrestam	Prelude and fugue, B flat major
L. Vierne	Fugue from 1st organ symphony
C.M. Widor	Fugue from 1st organ symphony
C.M. Widor	Fugue from 3rd organ symphony
C.M. Widor	Fugue from 4th organ symphony
H. Willan	Prelude and fugue, C minor (B146)
H. Willan	Prelude and fugue, B minor (B147)
H. Willan	Introduction, passacaglia and fugue (B149)
H. Willan	A fugal trilogy (B176)
E. Åkerberg	Prelude and fugue, G minor