On the Geometry of the Cross-Section of a Stem

Om stamtvärsnittets geometri

by

BERTIL MATÉRN

MEDDELANDEN FRÅN
STATENS SKOGSFORSKNINGSINSTITUT
BAND 46 · NR 11

CONTENTS

Introduction

All common methods of forest mensuration assume that the horizontal sections of the tree stems are circular. However, the cross-sections are never exactly circular. Many explanations have been suggested of this departure from circular form, see the very full account of the literature in Tischendorf 1943a.

The deviation from circular shape gives rise to errors in the assessment of cross-sectional areas, and hence in the calculation of volumes. In studying these errors several authors have investigated the case of *elliptical cross-sections*, or cross-sections composed of two semi-ellipses, see e.g. Chaturvedi 1926, Tischendorf 1927, 1943b, Heikkilä 1927, Tirén 1929, Stoffels 1948, and Matusita et al. 1955. Some of these papers also contain references to earlier investigations.

The justification offered for this approach is that most stems have different diameters in different directions, and that maximum and minimum diameters often intersect at approximately right angles. However, the author has not found in the literature any investigation of the real shape of cross-sections. The question whether an ellipse or two semi-ellipses can be regarded as a realistic model of the stem-section, must therefore be regarded as unsettled. The short-comings of the elliptic approach are emphasized in Tirén 1929 (see pp. 245, 248).

It therefore seems to be of interest to try to find out what statements about the errors of different mensurational methods that can be made without postulating anything about the shape of the cross-sections. Such a study may also give some hints as to how to carry out an empirical investigation on the form of the sections.

As indicated in Ch. I below, the appropriate starting point is *the theory of convex regions*. Fortunately many definite statements have been proved about convex regions, or to quote Blaschke (1920, p. 146): It is particularly remarkable that from the weak requirement of convexity there follows such a wealth of beautiful and profound conclusions ("Deshalb ist es besonders merkwürdig, dass sich aus der schwachen Forderung der Konvexität eine solche Fülle schöner und tiefliegender Folgerungen ziehen lässt"). It will be seen that some of the conclusions reached in the elliptic case are valid also under general conditions.

Since we shall restrict the study to those errors which arise from the geometric properties of the cross-section, *errors of measurement will be disregarded.*

In the main text, only the results of the mathematical treatment of the problem will be presented. The mathematical deductions will be found in an appendix. However, some of the geometric concepts seem to be of value for a discussion of the general aspects of the problem, and will hence be dealt with in the main text. We start with one such concept.

I. The convex closure of a region

The periphery of a stem section is seldom quite smooth, owing to bark ridges, fissures, etc. In addition, concave, or undulating, portions are sometimes met with, especially in the lower part of the stem. If a rubber-band is strapped around the stem, the area inside the band is therefore usually larger than the cross-sectional area. The contour formed by this rubber-band, or the region

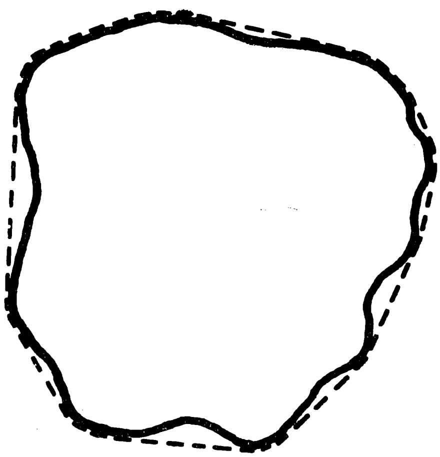

Fig. 1. The convex closure of a region.
Det konvexa höljet till en yta.

inside it, corresponds to what is called in geometry *the convex closure of a region,* see fig. 1. This concept seems to be useful in the present context, since readings of calipers and girth tapes are influenced only by the shape of the convex closure of the cross-section. The same holds true also of Bitterlich's "Winkelzahl"-method.

Every non-convex region has a smaller area and a greater perimeter than its convex closure. The difference between the area of the convex closure of a cross-section and the true cross-sectional area may be called *the convex deficit* of the cross-section. The area of the convex closure will in the sequel be denoted by g.

II. Girth measurements

On a tape, strapped around the stem, we read the perimeter, c (say), of the convex closure of the cross-section. We calculate the diameter, and the sectional area, from the following formulas.

$$\text{Diameter: } D_0 = c/\pi$$
$$\text{Area: } g_0 = c^2/4\pi$$

Thus g_0 is the area of a circle with perimeter c. Owing to the so-called "isoperimetric property" of the circle g_0 is greater than g, unless the convex closure of the cross-section is circular. The difference g_0—g is the so-called *isoperimetric deficit.*

III. Notation for diameters

For our purpose, it is convenient to define a diameter of the cross-section as the distance between two parallel tangents to the convex closure of the section. This corresponds to the reading of the common caliper. We thus have one diameter, $D(v)$, for every angle, v, which the bar of the caliper forms with a fixed direction in the plane, see fig. 2. We define *the* diameter of the cross-section as the arithmetic mean of $D(v)$, taken over all angles. Following a fundamental theorem, published by Cauchy in 1841 (see Blaschke 1936, p. 1), this mean diameter is equal to c/π, where c, as above, is the perimeter of the convex closure. For this ratio, we have already, in Ch. II, introduced the symbol D_0.

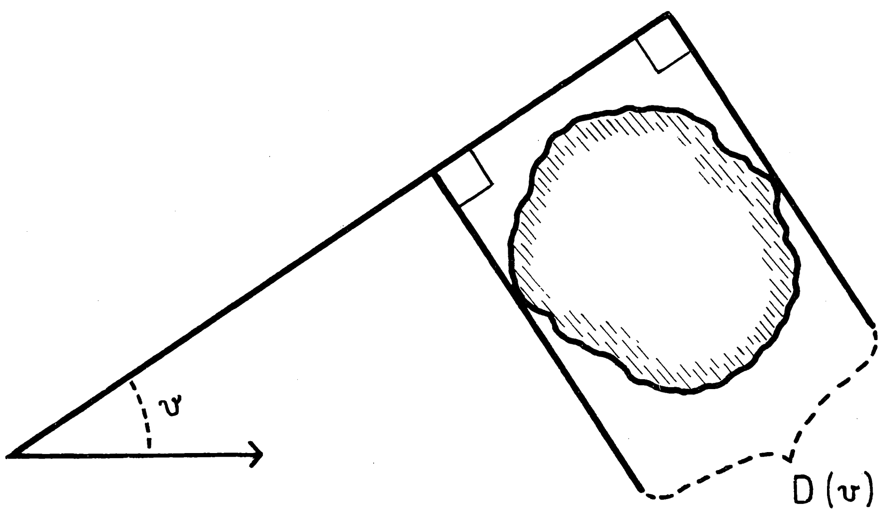

Fig. 2. A diameter defined as the distance between two parallel tangents.
En diameter definierad som avståndet mellan två parallella tangenter.

In passing, we note that the implications of Cauchy's theorem can also be formulated in the following manner: *If* $\bar{D}$ *is the arithmetic mean of the diameters of a great number of trees, calipered in random directions, then*—disregarding sampling errors, and errors of measurements—$\pi\bar{D}$ *equals the average circumference of the trees as obtained by tape.* This holds true irrespective of the form of the cross-sections.

For the *variance* of the different diameters, $D(v)$, of the cross-section, around their mean, D_0, we shall use the symbol σ^2. We also need a symbol for the *coefficient of correlation between two diameters taken at right angles.* We shall denote this coefficient by r.

IV. The orbiforms

There are cases in which all diameter measurements give the same value, which, according to Cauchy's theorem, then must be equal to D_0, the value obtained from girth measurement. This, of course, happens when the variance, σ^2, vanishes.

One might at first surmise that the convex closure then is a circle. However, the circle is only one member of a family of convex curves, characterized by the property that the distance between two parallel tangents is the same in all directions. For these curves the term *orbiform*[1] was proposed by Euler in 1778, see Tiercy 1920, cf. also Buchheim 1938.

Three examples of orbiforms are given in fig. 3. In 3a is shown the so-called Reuleaux-triangle, which is formed by three equal circular arcs, intersecting at 120°. The analytic definition of the curves 3b and 3c is found in the appendix. The isoperimetric deficits of the three curves amount to 11.4 % (3a), 4.2 % (3b), and 1.6 % (3c). The deficit is here expressed as a percentage of the true area. The Reuleaux-triangle is the orbiform with maximum isoperimetric deficit.

We now conclude: *Even if a stem section is convex and has the same diameter,* D_0, *in all directions, the sectional area may fall short of the corresponding circular area,* $\pi D_0^2/4$.

We can also form the following conclusion: *Even if the diameters of a cross-section are known in every direction, we can not in general draw exact inferences about the shape and area of the cross-section.*

[1] According to Strubecker 1955, p. 54, the term "Gleichdick" is used in German engineering.

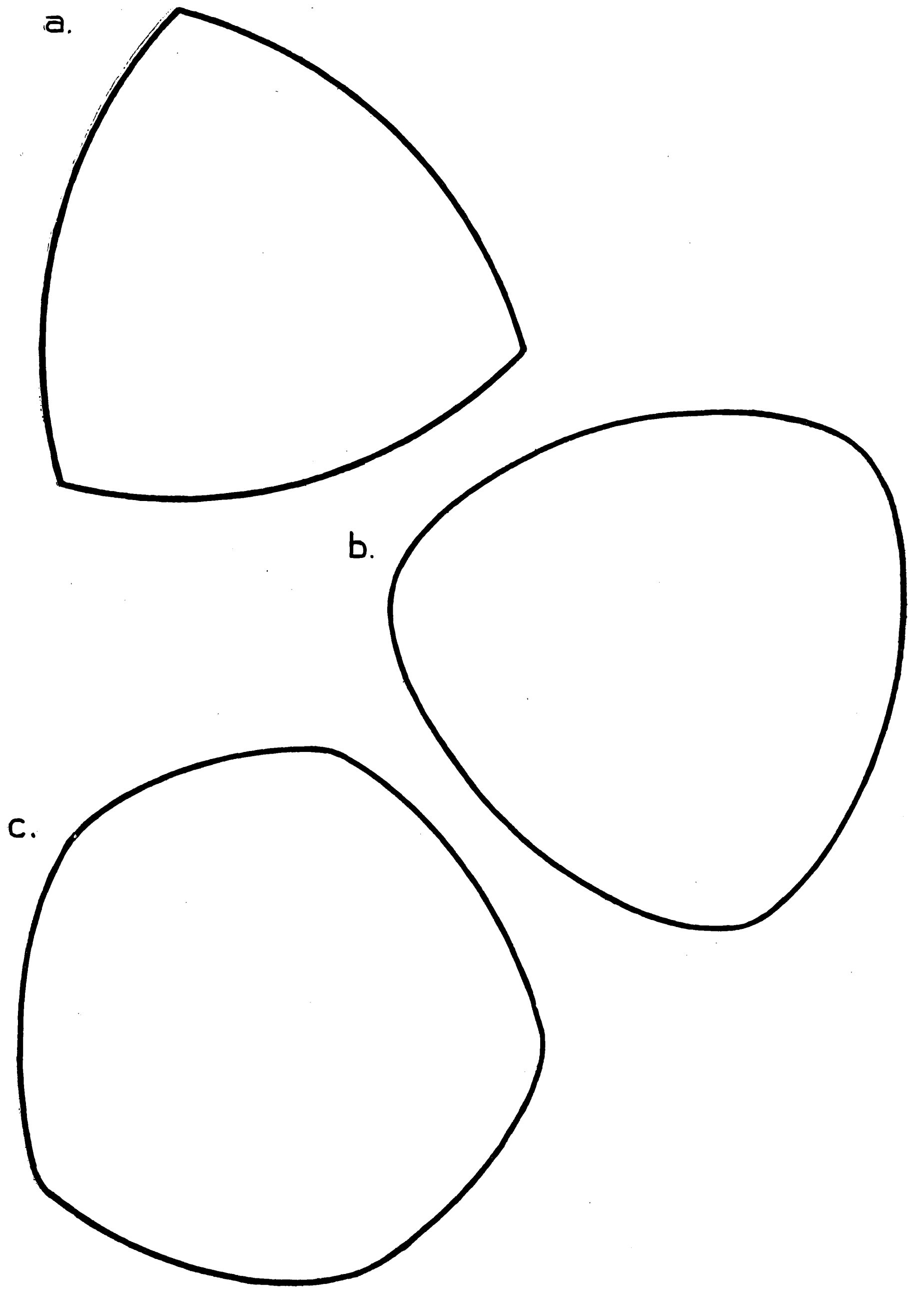

Fig. 3. Three examples of orbiforms.
Tre exempel på orbiformer.

V. Diameters calipered in random directions

We confine ourselves to estimates of the cross-sectional area, based on diameter measurements, of the form

$$\pi D^2/4$$

where D is a diameter calculated in one way or another from one or two caliperings. We here study the following four methods.

1) D is one diameter, calipered in a random direction.
2) D is the arithmetic mean of a diameter, $D(v)$, calipered in a random direction, and the diameter at right angles to $D(v)$.
3) Same as 2), but D is the geometric mean of the two diameters.
4) D is the geometric mean of two diameters, calipered in random, independently chosen, directions.

In this chapter only the *systematic errors* will be treated. We hence have to study *mathematical expectations of estimated areas*, i.e. averages over all possible directions. The expected values of the estimates of the sectional area obtained from the above four methods will be denoted by g_1, g_2, g_3, and g_4, respectively. The *sampling errors* will be dealt with in Ch. VII by way of numerical examples.

Using the symbols introduced in Ch. III, and applying well-known statistical formulas, we find:

$$g_1 = g_0 + \frac{\pi}{4}\sigma^2$$

$$g_2 = g_0 + \frac{\pi}{8}\sigma^2 (1 + r)$$

$$g_3 = g_0 + \frac{\pi}{4}\sigma^2 r$$

$$g_4 = g_0$$

Thus the fourth method is, on the average, equivalent to the estimation of cross-sectional area by girth measurement. It is not used in practice, and is included here only because of this equivalency.

We further note that g_2 is the arithmetic mean of g_1 and g_3.

In comparing g_1, g_2, and g_3, with one another, and with g_0 (or g_4), we need not bother with the case $\sigma^2 = 0$, already treated in Ch. IV. It is seen that the ranking of the methods is, to a certain extent, dependent on the value of r.

Now r, being a correlation coefficient, ranges from -1 to $+1$. We can therefore form the following tables of comparisons (for $\sigma^2 > 0$):

$r = -1$	$g_3 < g_0 = g_2 < g_1$
$-1 < r < 0$	$g_3 < g_0 < g_2 < g_1$
$r = 0$	$g_3 = g_0 < g_2 < g_1$
$0 < r < 1$	$g_0 < g_3 < g_2 < g_1$
$r = +1$	$g_0 < g_3 = g_2 = g_1$

We find that, on the whole, g_1 gives the highest overestimation, and that g_2 ranks next. If r is positive, also g_3 is higher than g_0. Now it must be remembered that, as shown in Chs. I and II, g_0 gives an overestimation equal to the "convex deficit" plus the "isoperimetric deficit".

However, if r is negative, the third method, i.e. using the geometric mean of diameters at right angles, gives on an average a lower value than g_0. The lowest possible value of g_3, for fixed σ, is attained for $r = -1$, and amounts to $g_0 - (\pi/4)\sigma^2$. Yet, g_3 can never be smaller than the area, g, of the convex closure. In the appendix (§ 3) we prove the following inequality

$$g_0 \geqq g + \frac{3\pi}{4}\sigma^2$$

which yields

$$g_3 \geqq g_0 - \frac{\pi}{4}\sigma^2 \geqq g + \frac{\pi}{2}\sigma^2$$

From this inequality we can also draw the following conclusion: *Irrespectively of the shape of the cross-section, the average difference between the "worst"* (g_1) *and the "best"* (g_0, g_4, *or* g_3) *estimate cannot be greater than the average difference between the "best" estimate and the area of the convex closure.* As is evident from the case of an orbiform, Ch. IV, the differences between the four methods may be considerably smaller than their common excess over the area of the convex region.

In the literature are found reports of very high overestimates of the area by girth measurements as compared to area estimates from calipered diameters, see Müller 1915, p. 82. This overestimation must be due to errors of measurement, errors of the instruments, or to subjective adjustment of the position of the instruments, cf. the discussion in Chaturvedi 1926, pp. 11 ff. See also Assmann 1956.

The case $r = -1$ deserves particular mention. In an ellipse of moderate excentricity r is very close to -1. E.g. for an ellipse with the ratio 0.8 between the axes, r is —0.9985. In Tirén 1929, numerical comparisons are made between the four methods in this case. The ensuing ranking of the methods is in close accordance with the above comparison for $r = -1$. (See also table 1, below). In the elliptic case the differences between the methods are relatively high in comparison with their excess over the true area.

On the other extreme we have the case $r = +1$. In this case methods 1, 2, and 3, are equivalent, whereas the girth tape gives a somewhat better value, cf. table 1.

In conclusion, we have found that the methods of this chapter (and of Ch. II) have a *positive bias*, unless the cross-section is exactly circular.

VI. Measuring maximum and minimum diameters

We now pass to the case where the sectional area is calculated from observations of the maximum and minimum diameters. By diameter we still understand the distance between two parallel tangents to the convex closure, i.e. the value obtained by calipering.

We now add the following cases to the previous list. In the formula $(\pi/4)D^2$, D is calculated as

5) the arithmetic mean of the maximum and minimum diameters,
6) the geometric mean of the maximum and minimum diameters,
7) the arithmetic mean of the maximum diameter and the diameter at right angles to the maximum diameter,
8) the geometric mean of the diameters in 7),
9) the arithmetic mean of the minimum diameter and the diameter at right angles to the minimum diameter,
10) the geometric mean of the diameters in 9).

It must be remarked that in some cases the estimates formed by methods 7)—10) are not well-defined. It may happen, e.g., that the maximum diameter is attained in two different directions, and that the corresponding diameters at right angles are unequal. In the subsequent illustrations, however, no such cases will appear.

One might have added some more methods to the above list, e.g. methods involving the quadratic mean of two observed diameters. However, the identity

$$\frac{D_1^2 + D_2^2}{2} - \left(\frac{D_1 + D_2}{2}\right)^2 = \left(\frac{D_1 + D_2}{2}\right)^2 - D_1 D_2$$

shows that areas calculated by using for D the quadratic mean can easily be obtained from the area estimates based on arithmetic and geometric means.

Returning to methods 5)—10), we note that if the convex closure is symmetric, e.g. an ellipse, methods 5, 7, and 9, are identical, as are methods 6, 8, and 10. In the elliptical case methods 6, 8, and 10, give the correct value of the area of the convex closure; methods 5, 7, and 9, give the arithmetic mean of g and g_1, see Ch. V. However, for other types of convex regions the six methods may all give different results. We denote the values of the cross-sectional areas, obtained from these methods, by $g_5, g_6, \ldots, g_{10}$, respectively.

The following inequalities follow from the fact that the geometric mean never is greater than the arithmetic mean

$$g_5 \geqq g_6, \qquad g_7 \geqq g_8, \qquad g_9 \geqq g_{10}$$

Further, we infer directly from the definitions of the methods

$$g_7 \geqq g_5 \geqq g_9, \qquad g_8 \geqq g_6 \geqq g_{10}$$

As is seen from the examples of Ch. VII, this list of inequalities is exhaustive in the general case. Further, it is not possible to find any general inequalities between the values obtained by the methods of this section, and the estimates dealt with in earlier chapters.

In contrast to the methods based on diameters calipered in random directions, *all the present methods may give underestimates of the area of the convex closure*, as is clear from the following example. A convex figure is composed of a square with side $\sqrt{2}$ and two opposite segments of the circle circumscribed around the square. In this case we have

$$g = 1 + \pi/2 = 2.57$$

$$g_7 = \frac{\pi}{4}\left(\frac{2+\sqrt{2}}{2}\right)^2 = 2.29$$

Thus, in this particular example

$$g_{10} = g_6 = g_8 < g_9 = g_5 = g_7 < g$$

We shall give some further comments on these methods in next chapter.

VII. Numerical illustrations

The methods presented above will now be applied to six different closed convex regions (or "ovals").

The regions are defined by their line supporting function (German: "Stützfunktion"), i.e. by a function, $p(v)$, giving the length of a perpendicular with slope v from a fixed interior point to the tangent of the contour, see fig. 4. The curves studied are the following

a) $$p(v) = \sqrt{100 \cos^2 v + 64 \sin^2 v}$$

Ellipse with semi-axes 8 and 10, used in Tirén 1929 for numerical illustrations.

b) $$p(v) = 9 + \cos 2v$$

Close resemblance with a), has the same maximum and minimum diameters.

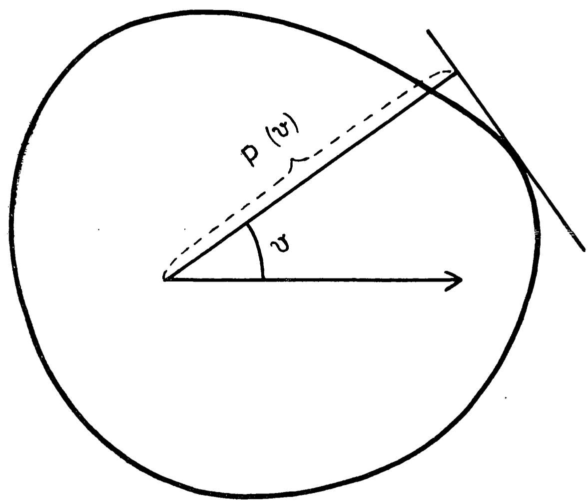

Fig. 4. The line supporting function, $p(v)$, of a convex region.
Stödfunktionen, $p(v)$, till en konvex yta.

c) $$p(v) = 16 + \cos 2v + \cos 3v$$

d) $$p(v) = 32 + 2 \cos 2v + \cos 3v + \cos 4v$$

Both c) and d) are "egg-shaped", resembling a curve composed of two semi-ellipses with different eccentricities.

e) $$p(v) = 35 + 2 \cos 2v + 2 \sin 4v$$

May be described as a "rounded rhomb". Maximum and minimum diameters do not intersect at right angles.

f) $$p(v) = 16 + \cos 4v$$

Has the shape of a "rounded square". As in the square, maximum and minimum diameters intersect at 45°.

The six curves are also shown in fig. 5.

Some further characteristics of the curves are found in table 1. The table also shows the estimated area, according to the methods of the previous chapters.

The six cases being only examples, no far-reaching conclusions can be drawn from table 1. However, it may be remarked that—in these examples—the

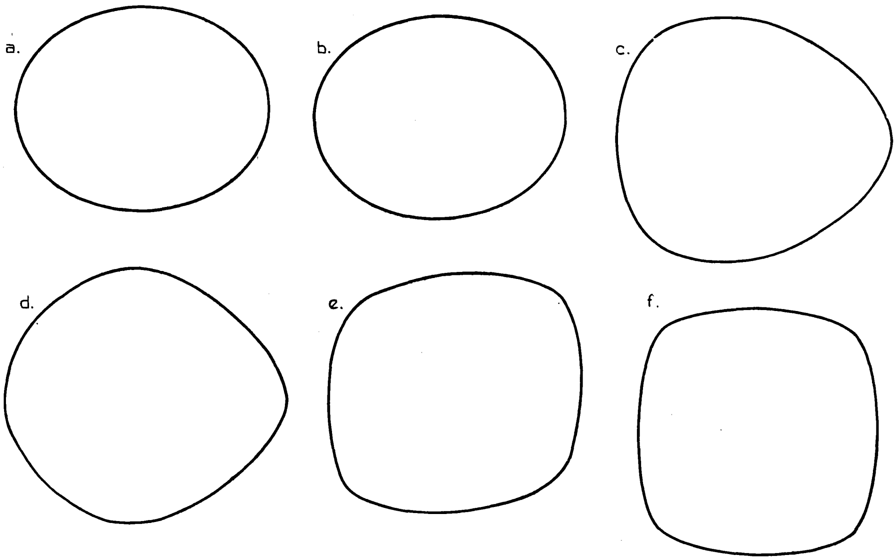

Fig. 5. Six examples of convex regions.
Sex exempel på konvexa ytor.

methods of Ch. II (girth measurement, g_0), and Ch. V (random diameters, $g_1, \ldots, g_4$), give more stable results than the methods of Ch. VI (maximum and minimum diameters, $g_5, \ldots, g_{10}$). As to the methods of Ch. V, it must be remembered that the figures in table 1 are "mathematical expectations" over all possible directions of the diameters calipered.

In the table no orbiform (Ch. IV) is included since all twelve methods are equivalent in this case—without necessarily giving a good estimate of the convex area.

As seen from the above definitions, the function $p(v)$ in examples *b*—*f* is a sum of trigonometric functions. In § 3 of the appendix we show that the supporting function of any closed convex curve can be represented approximately by a sum of this type.

The six examples will now be used to illustrate the *sampling errors*, connected with some of the methods. These errors appear in methods involving a random

Table 1. Six examples of convex regions. Characteristics of the region. Estimates of the area by different methods.

Example	a	b	c	d	e	f
Coefficient of variation (100 σ/D_0) . . .	7.82	7.86	4.42	4.94	5.71	4.42
Minimum diameter/ maximum diameter	0.800	0.800	0.882	0.871	0.817	0.882
Coeff. of correlation between diameters at right angles (r).	— 0.9985	— 1.0000	— 1.0000	— 0.6000	0.0000	1.0000
Area estimate in permille of true area:						
g_0 (= g_4)	1019	1019	1022	1017	1030	1030
g_1	1025	1025	1024	1020	1034	1032
g_2	1019	1019	1022	1018	1032	1032
g_3	1013	1013	1020	1016	1030	1032
g_5	1012	1019	1022	1066	1030	1030
g_6	1000	1006	1018	1061	1020	1026
g_7	1012	1019	1022	1082	1146	1163
g_8	1000	1006	1018	1078	1144	1163
g_9	1012	1019	1022	986	921	905
g_{10}	1000	1006	1018	985	919	905
Highest value is obtained by method.	1	1	1	7	7	7, 8
Lowest value is obtained by method.	6, 8, 10	6, 8, 10	6, 8, 10	10	10	9, 10

choice of the direction of the diameters to be measured. Hence they arise only in the methods of Ch. V, whereas girth measurement (Ch. II), and measurements of maximum and minimum diameters (Ch. VI) are unaffected by sampling errors.

It is evident that no sampling errors are present if the convex closure is an orbiform (Ch. IV) since then any calipering gives the value D_0.

We express the magnitude of the sampling errors by the *standard errors* of the different area estimates. The standard errors of the estimates 1—4 of Ch. V will be denoted σ_1, σ_2, σ_3, and σ_4, respectively.

The values of these standard errors in the six examples are found in table 2. To make the figures comparable with those of table 1, the standard errors are expressed in permille of the true area, g, of the convex region.

These standard errors obviously depend on the two quantities, σ and r, characterizing the variation of diameters in different directions. The following

Table 2. Six examples of convex regions. Standard errors (in permille of true area) of different estimates of the area.

Example	a	b	c	d	e	f
σ_1	159	160	90	101	118	91
σ_2	4	0	0	45	83	91
σ_3	9	4	1	44	83	91
σ_4	113	113	64	71	83	64

approximate relations, derived by the general method given in Cramér 1945, § 27.7, are seen to give values in close accordance with those of table 2.

$$\sigma_1/g_0 \approx 2\,\sigma/D_0$$

$$\sigma_2 \approx \sigma_3 \approx \sigma_1 \sqrt{\frac{1 + r}{2}}$$

$$\sigma_4 \approx \frac{\sigma_1}{\sqrt{2}}$$

VIII. Bitterlich's "Winkelzahl"-method

The "Winkelzahl"-method of estimating the basal area of a stand, has received well-deserved attention since its first publication (Bitterlich 1948): see e.g. Keen 1950, Grosenbaugh 1952, Seip 1952.

In studying the method, we shall confine ourselves to the case where the ground level is horizontal. Then all stem sections at breast height are in the same horizontal plane, H.

In applying the method, all stems are counted whose sections at breast height subtend an angle greater than or equal to a fix angle α when viewed from a randomly chosen point in H. The stem count may be repeated on several sample points. The average number of stems from such counts furnishes—after multiplication with the "Zählfaktor", $(100 \sin \alpha/2)^2$—an estimate of the sectional area at breast height, or basal area, of the stand in square meters per hectare. For small α, the factor can be taken as $(50\alpha)^2$.

Let G_α be the area of the region in H, from which a particular stem is seen under an angle $\geqq \alpha$. We suppose G_α defined so as to include the basal area of the stem. We then form the product

$$g(\alpha) = G_\alpha \sin^2 (\alpha/2)$$

The "Winkelzahl"-method is based on the fact that, if the cross-section is circular, $g(\alpha)$ equals g, the true basal area of the stem. Any bias of the method arising from a non-circular form of the cross-section at breast height will manifest itself in a deviation of $g(\alpha)$ from g.

Now, as indicated in § 4 of the appendix, in passing to the limit we get $g(0) = g_1$, where g_1 is the average basal area obtained from one calipering in a randomly selected direction (Ch. V). For small values of α, $g(\alpha)$ is very close to g_1. Since in the applications α is of the order of magnitude 1°, we infer: *As far as the deviation from circular form of the cross-sections is concerned, the "Winkelzahl"-method gives the same bias as the method of calipering every stem in one randomly chosen direction.* However, it must be observed that this conclusion is reached on the assumption that the ground level is horizontal.

It may finally be mentioned that a related method, proposed by Masuyama 1953, gives an unbiased estimate of the basal area under the sole condition that the cross-sections at breast height are convex. Masuyama's method is derived from Steiner's formula for the area of a curve parallel to a given convex curve (see Blaschke 1936, formula 150, p. 26).

IX. A remark about some related problems in forest mensuration

The difficulties encountered in calculating the sectional area of a stem arise from the circumstance that the cross-section is observed "from the outside". To put this in other words, we can attribute the complications to the fact that diameters—in the sense of Ch. III—and not radii are measured.

If a region, convex or not, has uniquely determined radii from a fix interior point, and these radii can be measured, then an unbiased estimate of the area is available. It consists of the simple expression πr^2, where r is the length of a radius with random direction (cf. appendix, § 5). If several radii, e.g. r_1, r_2, r_3, and r_4, are measured, the unbiased estimate takes the form

$$\frac{\pi}{4}(r_1^2 + r_2^2 + r_3^2 + r_4^2)$$

Thus, if this expression is written as πr^2, r shall be the quadratic mean of the four radii.

This remark pertains to measurements of the area of end-sections of logs, estimation of the area of the crown projection of a tree, etc.

X. Summary and discussion

The common methods of forest mensuration assume that the cross-sections of tree stems are circular. However, the sections always depart more or less from circular form. This deviation gives rise to errors in the assessment of cross-sectional areas. In the present investigation the author has tried to apply geometric concepts and theorems of a universal nature and derive from them general statements about the performance of mensurational methods. He is well aware that a mathematician, more acquainted with the relevant branches of geometry, might elicit a richer fund of statements.

When calipering a stem, or measuring the girth, or performing other similar observations "from the outside", e.g. making a stem count according to Bitterlich's method, only the *convex closure* of the section is involved: see figs. 1 and 2. The true sectional area is in general smaller than the area of the convex closure, it has a *"convex deficit"*.

The so-called *"isoperimetric deficit"* results from the fact that the area of the convex closure is smaller than the area of a circle with the same perimeter as the convex closure. An isoperimetric deficit is usually present even in cases in which the diameter is constant in all directions. This is due to the fact that the circle is only one of the *orbiforms*, or curves with constant diameter, see fig. 3.

The above concepts are introduced in Chs. I, and IV. In Chs. II, III, and V, a study is made of some common methods of estimating the cross-sectional area, viz. by girth-measurement and by calipering diameters in randomly selected directions. It is found that those methods, except in the circular case, have a positive bias in comparison with the true area of the convex closure. If the convex regions are divided into two main types, it is possible for each one of these types to rank the different methods according to the magnitude of the bias. Except in some special cases where the convex closure is "ellipse-like", the methods give average values that are close to one another in comparison with their excess over the area of the convex closure.

Methods involving maximum and/or minimum diameters seem to be less stable, in the sense that they are more dependent upon the assumptions made on the shape of the cross-section. They are dealt with in Ch. VI.

By expanding the line-supporting function [$p(v)$ in fig. 4] of a convex region in Fourier-series, some general inequalities can be obtained between different area estimates. In this way numerical expressions of convex regions can also be obtained, Ch. VII. Fig. 5 shows some examples, which in a schematic—and perhaps exaggerated—way show various types of departure from circular form. The examples are also used to illustrate the sampling errors connected with those methods which involve a random choice of the direction of the diameters to be calipered.

Bitterlich's "Winkelzahl"-method, dealt with in Ch. VIII, is dependent upon the assumption of circular cross-sections at breast height. In this respect, however, it is equivalent to the method of estimating the basal area by calipering one randomly selected diameter of every stem.

If a region, more or less close to circular shape, is observed "from the inside" by measuring the radii, the bias present in measurements "from the outside" can be avoided, Ch. IX.

The "geometry of cross-sections" can provide only part of the information necessary to survey the performance of mensurational methods. Measuring a physical quantity, such as a cross-sectional area, we have to face the difficulty of making exact definitions, the influence of imperfect instruments, the bias of subjective judgments, etc.[1]. The following conclusions must therefore be regarded merely as tentative expressions of the views of the author. They may be of interest in cases, when we require high accuracy. In such cases the geometry of the situation will play a more important role, since we try to reduce the influence of other disturbing factors as much as possible.

In estimating the cross-sectional area from two diameter caliperings the arithmetic mean or the geometric mean of the two diameters may be used without any difference of great practical importance.

The estimates based on maximum and minimum diameters cannot be recommended without a thorough investigation of the actual shape of the cross-sections of stems.

The geometric study gives some support to the use of girth tape instead of caliper. However, a discussion of tape versus caliper must involve many other considerations. It would therefore be premature to try to come to a definite conclusion in this controversial question.

In order to get an idea of the actual magnitude of the errors here discussed, a sample of sawn tree sections should be inspected. Observations should be made in a very accurate way, so that errors of measurement could be separated from errors arising from the geometrical properties of the stem sections. By carrying out the measurements in terms of the convex deficit, the isoperimetric deficit, the diameter variance, etc., a good understanding would be obtained of the ways in which the geometry of cross-sections influences the performance of mensurational methods.

The author is indebted to professor Lars Tirén who has kindly read the manuscript and made valuable comments. He also wishes to express his gratitude to Miss Greta Nilsson for performing the calculations, and to Mrs. Anneliese Neuschel for drawing the figures. Finally, acknowledgment is due to Dr. John T. Lewis of the Mathematical Institute at the University of Oxford, who has corrected the English of the paper.

Appendix

§ 1. Some general formulas for a closed convex region

Let the origin be an interior point of the bounded closed convex region C. The line supporting function, $p(v)$, is defined as shown in fig. 4.

[1] For a discussion of such factors, see the fundamental paper by Tirén (1929).

Assuming $p(v)$ to be twice differentiable, the condition for convexity is

$$p(v) + p''(v) \geqq 0 \quad \text{for all } v \tag{1}$$

For the rectangular coordinates of the contour of C, we get the following parametric representation

$$x = p(v) \cos v - p'(v) \sin v$$

$$y = p(v) \sin v + p'(v) \cos v$$

The diameter in direction v:

$$D(v) = p(v) + p\,(v + \pi) \tag{2}$$

The (mean) diameter of C (Cf. Ch. III):

$$D_0 = \frac{1}{\pi}\int_0^{\pi} D(v)\,dv = \frac{1}{\pi}\int_0^{2\pi} p(v)\,dv \tag{3}$$

The area of C:

$$g = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\,dv \tag{4}$$

For the perimeter, c, of C, we have Cauchy's formula, referred to in Ch. III:

$$c = \int_0^{2\pi} p(v)\,dv = \int_0^{\pi} D(v)\,dv = \pi D_0 \tag{5}$$

The variance of the diameters of C is

$$\sigma^2 = \frac{1}{\pi}\int_0^{\pi} (D - D_0)^2\,dv \tag{6}$$

The coefficient of correlation, r_α, between diameters intersecting at an angle α, is found from the formula

$$\sigma^2 r_\alpha = \frac{1}{\pi}\int_0^{\pi} [D(v) - D_0]\,[D(v + \alpha) - D_0]\,dv \tag{7}$$

For $r_{\pi/2}$ the symbol r is used for brevity in Chs. III and V.

By means of (2) we can express D_0, σ^2, and r, as functionals of $p(v)$.

The fundamental formulas on the line supporting function are found in textbooks on differential geometry. We note that the formulas (2)—(7) are valid also in the case in which $p'(v)$ has a finite number of discontinuities.

§ 2. Examples of closed convex regions

Various types of closed convex regions can be obtained by choosing for $p(v)$ different functions of period 2π. In Chs. IV and VII, we have used expressions of the form

$$p(v) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kv + A_k) \tag{8}$$

where only a finite number of the a_k's are $\neq 0$, and the constants are so chosen that formula (1) is satisfied.

In this case we get

$$D(v) = 2\,a_0 + 2 \sum_{1}^{\infty} a_{2k} \cos(2\,kv + A_{2k}) \tag{9}$$

If all a_k of even order, except a_0, are 0, we hence get an orbiform, Ch. III. The two orbiforms b, and c, in fig. 3, have the equations

$$p(v) = 10 + \cos 3\,v \tag{3 b}$$

$$p(v) = 32 + \cos\left(3\,v + \frac{\pi}{4}\right) + \cos 5\,v \tag{3 c}$$

The rectangular coordinates of the contour can be obtained from

$$x = a_0 \cos v + \frac{1}{2} \sum_{1}^{\infty} a_k \left\{(k+1)\cos[(k-1)\,v + A_k] - (k-1)\cos[(k+1)\,v + A_k]\right\}$$

$$y = a_0 \sin v - \frac{1}{2} \sum_{1}^{\infty} a_k \left\{(k+1)\sin[(k-1)\,v + A_k] + (k-1)\sin[(k+1)\,v + A_k]\right\}$$

Inserting (8), or (9), in formulas (3)—(7), we find

$$D_0 = 2\,a_0 \tag{10}$$

$$g = \pi\,a_0^2 - \frac{\pi}{2} \sum_{2}^{\infty} (k^2 - 1)\,a_k^2 \tag{11}$$

$$c = 2\,\pi\,a_0 \tag{12}$$

$$\sigma^2 = 2\sum_1^\infty a_{2k}^2 \tag{13}$$

$$\sigma^2 r_\alpha = 2\sum_1^\infty a_{2k}^2 \cos(2\,k\,\alpha) \tag{14}$$

Hence, for $\alpha = \pi/2$:

$$r = \sum_1^\infty a_{2k}^2\,(-1)^k \Big/ \sum_1^\infty a_{2k}^2 \tag{15}$$

From (15) is seen that we can, for every $\sigma^2 > 0$, construct curves with any desired value of r.

Substituting these expressions in the formulas of Chs. II, and V, we can express $g_0, \ldots, g_4$ in the coefficients of (8).

The area estimates of Ch. VI, however, must be separately evaluated in each case, by solving trigonometric equations. In the examples of fig. 5, these equations are easy to solve, but they are very complicated in the general case.

In evaluating the standard errors of the area estimates of Ch. V, the estimated area is first expressed as a linear function of trigonometric expressions.

To take an example, the area estimate of example b in Ch. VII, according to method 3, is

$$\pi\,(9 + \cos 2\,v)\,(9 - \cos 2\,v) = \frac{\pi}{2}\,(161 - \cos 4\,v)$$

The error variance, $\sigma_3{}^2$, is then obtained in the same way as σ^2 in (13) is derived from (9).

In what concerns the numerical treatment of the elliptic case—example a of Ch. VII—the reader is referred to Tirén 1929.

§ 3. Proof of an inequality in Ch. V

Any bounded, closed convex region in the plane can be approximated by a convex polygon in such a way that the area, g, the perimeter, c, the diameter variance, σ^2, and all other characteristics appearing in the left hand sides of formulas (3)—(7), are arbitrarily close to the corresponding characteristics of the given region. Thus, all equations, and inequalities, involving these quantities, that are true for a convex polygon, will hold in the general case.

Now, for a polygon, $p'(v)$ has a finite number of discontinuities, and is of bounded variation. Thus, in this case, formulas (3)—(7) are valid. Further, by Jordan's test (Titchmarsh 1939, § 13.232), $p'(v)$ can be developed in a Fourier series, converging everywhere to $\frac{1}{2}[p'(v-0) + p'(v+0)]$. By termwise integration we obtain a Fourier-expansion of $p(v)$, valid everywhere. We choose the notation so that this series is the one of formula (8), now interpreted as an infinite series. Then integrals of the type

$$\int_0^{2\pi} p(v)\, p'(v + \alpha)\, dv, \qquad \int_0^{2\pi} [p'(v)]^2\, dv$$

etc., can be expressed in the constants of series (8), by means of Parseval's theorem. Therefore formulas (10)—(15) are valid also for a convex polygon.

Using (11), (12), (13), and the definition $g_0 = c^2/4\pi$ of Ch. II, we get

$$g_0 - \frac{3\pi}{4}\sigma^2 - g = 2\pi \sum_1^{\infty} (k^2 - 1)\, a_{2k}^2 + 2\pi \sum_1^{\infty} k(k+1)\, a_{2k+1}^2$$

Here, all expressions on the right hand side are non-negative. We consequently get the inequality of Ch. V:

$$g_0 \geqq g + \frac{3\pi}{4}\sigma^2 \qquad (16)$$

The sign of equality holds in (16) only if $p(v)$ is of the type exemplified by the "ellipse-like" curve of fig. 5b, i.e. if all a_k with $k > 2$ vanish.

Several inequalities concerning the isoperimetric deficit, the area of a convex region with given maximum and minimum diameters etc., are to be found in the literature. See e.g. Bonnesen & Fenchel 1934, pp. 74 ff., Santaló 1953, pp. 37 ff.

§ 4. Formulas for Bitterlich's method

In studying Bitterlich's method, we first need an expression of the area, G_α in the plane of C, from which C is seen under an angle $\geqq \alpha$ (see Ch. VIII).

Two tangents, whose perpendiculars from the origin have slopes v and $\pi + v - \alpha$, intersect in a point with coordinates

$$x = (-1/\sin\alpha)\,[p(v)\sin(v-\alpha) + p(v+\pi-\alpha)\sin v]$$
$$y = (1/\sin\alpha)\,[p(v)\cos(v-\alpha) + p(v+\pi-\alpha)\cos v]$$

Hence, after straightforward calculations:

$$G_\alpha = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, dv =$$

$$= (1/\sin^2\alpha) \int_0^{2\pi} p(v)\,[p(v) + \cos\alpha \cdot p(v+\pi-\alpha) + \sin\alpha \cdot p'(v+\pi-\alpha)]\, dv \qquad (17)$$

By this formula G_α includes also the area of C.

We then pass to the product

$$g(\alpha) = \sin^2(\alpha/2) \cdot G_\alpha \qquad (18)$$

introduced in Ch. VIII. If C is a circle, $g(\alpha)$ is, for all values of α, equal to the area, g, of C. In the general case, geometrical considerations give the following limiting values:

$$g(0) = g_1 \qquad g(\pi) = g \qquad (19)$$

where, as before, g_1 is the average value obtained by calipering in a random direction. The mean value of $g(\alpha)$, when α varies from 0 to π, can be found from a formula of Crofton (see Santaló 1953, formula 4.6, p. 21)

$$\pi g + \iint (\alpha - \sin\alpha)\, dx\, dy = c^2/2 \qquad (20)$$

Here α is the angle under which C is seen from the point (x, y). The integration is taken over all points outside C. Substituting G_α in (20), the formula can be written

$$\pi g + \int_0^\pi (\sin\alpha - \alpha)\, dG_\alpha = c^2/2$$

Using (18) and (19), we find

$$\lim_{\alpha \to 0} (\sin\alpha - \alpha) \cdot G_\alpha = g_1 \lim\, [(\sin\alpha - \alpha)/\sin^2(\alpha/2)] = 0$$

Hence by partial integration:

$$\int_0^\pi G_\alpha (1 - \cos\alpha)\, d\alpha = c^2/2$$

or

$$\frac{1}{\pi} \int_0^\pi g(\alpha)\, d\alpha = c^2/4\pi = g_0 \qquad (21)$$

where g_0 is the estimate of area obtained by girth measurement (Ch. II).

Formulas (19) and (21) give some information about the course of $g(\alpha)$ as a function of α. In fig. 6, $g(\alpha)$ is shown for the three regions in figures 3b, 5b, and 5f. As seen from fig. 6, for small values of α, $g(\alpha)$ may be either smaller or greater than g_1.

When $p(v)$ is given by (8) we find for $g(\alpha)$ the expression

$$g(\alpha) = g_0 + \frac{\pi}{2} \sum_{2}^{\infty} a_k^2 \frac{1 + \cos\alpha \cdot \cos(k\alpha + k\pi) + k \sin\alpha \cdot \sin(k\alpha + k\pi)}{1 + \cos\alpha} \qquad (22)$$

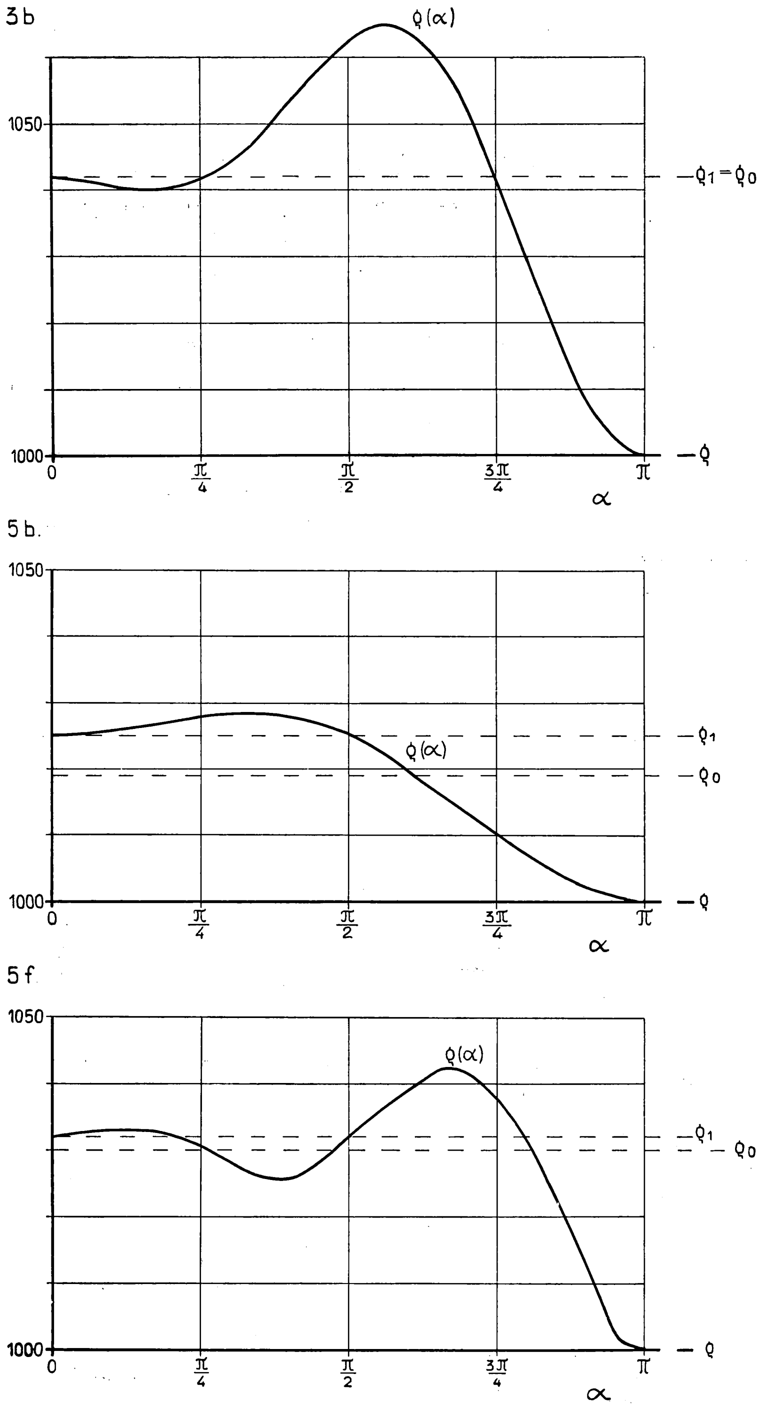

Fig. 6. The function $g(\alpha)$ for the convex regions of figures 3b, 5b, and 5f.

By somewhat lengthy, but trivial calculations, we can derive from (22) the following inequalities:

$$| g(\alpha) - g_0 | \leqq g_0 - g \tag{23}$$

$$| g(\alpha) - g_1 | \leqq (g_1 - g) \operatorname{tg}^2 (\alpha/2) \tag{24}$$

From (23) is seen that $g(\alpha)$ can never be smaller than the true area, g, of the region C. From (24) we conclude that for such small values of α, as used in practice, $g(\alpha)$ is very close to g_1, as stated in Ch. VIII.

§ 5. Measuring the area "from the inside"

The statement of Ch. IX is a direct consequence of the formula for the area of a region, expressed in polar coordinates.

Suppose that the origin is an interior point of the bounded region C, and assume further that, for every v, the radius with direction v intersects the boundary of the region in one point, with distance $r(v)$ from the origin. Then the area of the region is

$$\frac{1}{2} \int_0^{2\pi} r^2(v)\, dv$$

as shown in textbooks on integral calculus. To put this in words: the area is equal to the squared "quadratic mean" of the radii, multiplied by π.

References

Assmann, E., 1956: Wie kann der Grundflächenzuwachs auf Dauer-Versuchsflächen genauer bestimmt werden? — Paper presented at the 12th Congress of the International Union of Forest Research Organizations, Oxford 1956.

Bitterlich, W., 1948: Die Winkelzählprobe. — Allgemeine Forst- und Holzwirtschaftliche Zeitung, 59: 4—5.

Blaschke, W., 1920: Kreis und Kugel. — Leipzig.

— 1936: Vorlesungen über Integralgeometrie. I. Zweite Aufl. — Hamburg.

Bonnesen, T. & Fenchel, W., 1934: Theorie der konvexen Körper. — Berlin. (Ergebnisse der Mathematik und ihrer Grenzgebiete, III: 1.)

Buchheim, W., 1938: Kluppung und Kreisform des Stammquerschnittes. Über Stammquerschnitte gleicher Breite in allen Richtungen. — Zeitschrift für Forst- und Jagdwesen, 70: 656—658.

Chaturvedi, M. D., 1926: Measurements of the cubical contents of forest crops. — Oxford Forestry Memoirs, 4.

Cramér, H., 1945: Mathematical methods of statistics. — Uppsala.

Grosenbaugh, L. R., 1952: Plotless timber estimates. New, fast, easy. — Journal of Forestry, 50: 32—37.

Heikkilä, T., 1927: Über die Ermittelung der Querfläche eines Stammes. — Acta forestalia fennica, 32: 3.

Keen, E. A., 1950: The relascope. — Empire Forestry Review, 29: 253—264.

Masuyama, M., 1953: A rapid method of estimating basal area in timber survey—an application of integral geometry to areal sampling problems. — Sankhyā, The Indian Journal of Statistics, 12: 291—302.

Matusita, K., et al., 1955: Some problems of sampling in the forest survey. — Annals of the Institute of Statistical Mathematics, Tokyo, 7: 1—23.
Müller, U., 1915: Lehrbuch der Holzmesskunde. Zweite Aufl. — Berlin.
Santaló, L. A., 1953: Introduction to integral geometry. — Paris. (Actualités scientifiques et industrielles, 1198.)
Seip, H. K., 1952: Relaskopet. — Tidsskrift for Skogbruk, 60: 22—26.
Stoffels, A., 1948: De berekening van het totale grondvlak, — — — (The determination of the total basal area, — — —)—Nederlandsch Boschbouw-Tijdschrift, 20: 331—341, 352—358.
Strubecker, K., 1955: Differentialgeometrie. I. Kurventheorie der Ebene und des Raumes. — Berlin. (Sammlung Göschen, Band 1113/1113a.)
Tiercy, G., 1920: Sur les courbes orbiformes. Leurs utilisation en mécanique. — Tôhoku Mathematical Journal, 18: 90—115.
Tirén, L., 1929: Über Grundflächenberechnung und ihre Genauigkeit. — Meddelanden från Statens skogsförsöksanstalt, 25: 5.
Tischendorf, W., 1927: Lehrbuch der Holzmassenermittlung. — Berlin.
— 1943a: Über Gesetzmässigkeit und Ursache der Exzentrizität von Baumquerflächen. — Centralblatt für das gesamte Forstwesen, 69: 33—54.
— 1943b: Der Einfluss der Exzentrizität der Schaftquerflächen auf das Messungsergebnis bei Bestandesmassenermittlungen durch Kluppung. — Centralblatt für das gesamte Forstwesen, 69: 87—94.
Titchmarsh, E. C., 1939: The theory of functions. Second edition. — Oxford.

Sammanfattning

Om stamtvärsnittets geometri

De vanliga metoderna för uppskattning av trädstammens volym och grundyta förutsätter att stammens tvärsnitt är cirkelformigt. Det fel, som uppkommer därigenom att tvärsnittet i verkligheten avviker från cirkelformen, måste beaktas vid sådana mätningar, vid vilka hög noggrannhet eftersträvas.

Vid klavning av en stam, vid omkretsmätning med måttband och andra »utanpå»-mätningar, t. ex. stamräkning enligt Bitterlichs metod, har man kontakt endast med stamtvärsnittets *»konvexa hölje»*, se fig. 1 och 2. Tvärsnittets verkliga yta är i allmänhet mindre än det konvexa höljets, har ett *»konvext deficit»*.

Det s. k. *»isoperimetriska deficitet»* uppkommer därigenom att det konvexa höljets yta i allmänhet är mindre än ytan av en cirkel med samma omkrets. Ett isoperimetriskt deficit — på upp till 11,4 % av den verkliga ytan — kan finnas, även om tvärsnittet är konvext och den klavade diametern är densamma i alla riktningar. Cirkeln är nämligen endast en av de s. k. *orbiformerna*, ytor med konstant diameter. Några ex. på dylika ytor återfinns i fig. 3.

Med utgångspunkt i en mätning av omkretsen, c, uppskattar man tvärsnittets yta till $g_0 = c^2/4\pi$, som är lika med den verkliga ytan plus det konvexa och det isoperimetriska deficitet. Denna yta kan även skrivas $g_0 = \pi D_0^2/4$, där D_0 är det aritmetiska medeltalet av diametrar i alla riktningar.

Vid klavning av en diameter, D, med på måfå vald riktning är motsvarande skattning av tvärsnittets yta, $\pi D^2/4$. Denna skattning är i genomsnitt större än den genom omkretsmätning erhållna ytan, g_0. (Här liksom eljest i uppsatsen bortses från mätningsfel.) Skillnaden beror på variationen mellan tvärsnittets diametrar i olika riktningar. Om D i det nämnda uttrycket för ytan är det aritmetiska medeltalet av en på måfå vald diameter och en mot denna vinkelrät diameter, får man i genomsnitt också en, om än något mindre, överskattning i förhållande till g_0. Om slutligen D är det geometriska medeltalet till två sådana diametrar, kan man i genomsnitt få ett högre värde än g_0, eller ett lägre, beroende på det konvexa höljets form. Om man bortser från vissa speciella fall, då det konvexa höljet till formen liknar en ellips, ger de nu nämnda metoderna genomsnittliga värden, som ligger tämligen nära varandra, jämfört med deras avvikelse från det konvexa höljets yta.

De metoder, som bygger på klavning av maximi- och minimidiametrar, eller en av dem och en mot denna vinkelrät diameter, synes vara i högre grad beroende av antaganden om tvärsnittets form. De kan i vissa fall ge relativt stora överskattningar i förhållande till det konvexa höljets yta, i andra fall rätt kraftiga underskattningar.

Genom att uttrycka ett konvext områdes »stödfunktion» [$p(v)$ i fig. 4] i s. k. Fourier-serie kan man visa vissa allmänna satser om de nu berörda metoderna för skattning av tvärsnittets yta. På så vis kan man även få numeriska exempel på konvexa ytor av olika form, se fig. 5.

Bitterlichs »Winkelzahl»-metod är i fråga om det systematiska fel, som uppkommer genom att stammens grundyta inte har cirkelform, jämställd med grundyteuppskattning genom en klavning i slumpmässigt vald riktning. Detta har dock visats endast under förutsättning att marken inte lutar.

Om mer eller mindre cirkelliknande ytor kan observeras »inuti», genom radiemätningar, kan man undvika de systematiska fel, som uppkommer vid »utanpåmätning». Ex.: mätning av en stocks ändytor, en trädkronas projektion.

Uppskattningen av stamtvärsnittets yta har i denna utredning setts endast från rent geometrisk synpunkt. Några bestämda rekommendationer om lämpliga uppskattningsförfaranden kan därför inte lämnas. Nedanstående antydningar får därför endast fattas som uttryck för en rent subjektiv bedömning av situationen.

När tvärsnittets yta skall uppskattas med hjälp av två klavade diametrar, spelar det ur praktisk synpunkt en rätt liten roll om man använder det aritmetiska eller det geometriska medeltalet till de två diametrarna.

Uppskattningar grundade på högkants- och lågkantsmätning kan inte rekommenderas utan en grundlig undersökning av den faktiska formen hos trädens stamtvärsnitt.

Den geometriska undersökningen ger ett visst stöd för användningen av omkretsmätning i stället för klavning. En mängd andra omständigheter spelar emellertid in vid jämförelsen mellan måttband och klave, varför det vore förhastat att här draga en bestämd slutsats i denna omstridda fråga.

Utredningen synes kunna giva vissa hållpunkter för planläggningen av ett empiriskt studium av stamtvärsnitt. Om man på en noggrant vald samling av stamtrissor gör observationer av det konvexa deficitet, det isoperimetriska deficitet, diametervariansen etc., bör man få en rätt god inblick i hur »stamtvärsnittets geometri› påverkar olika i praktiken tillämpade mätnings- och uppskattningsförfaranden.