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On the Estimation of the Distribution of Sample Means Based on Non-Stationary Spatial Data

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ABSTRACT. Two different methods of estimating the distribution of sample means based on non-stationary spatially indexed data $\{X_{\boldsymbol{i}} : \boldsymbol{i} \in \mathcal{I}\}$, where $\mathcal{I}$ is a finite subset of the integer lattice $\mathbb{Z}^2$, are presented. The information in the different cells in the lattice are allowed to come from different distributions, but with the same expected value or with expected values that can be decomposed additively into directional components. Furthermore, neighboring lattice cells are assumed to be dependent, and the dependence structure is allowed to differ over the lattice. It will be shown under such rather general conditions that the distribution of the sample mean can be estimated by resampling, as well as by a normal approximation for which a non-parametric estimator of variance is provided. The developed methods can be applied in assessing accuracy of statistical inference for spatial data.

Key words: resampling, $\boldsymbol{m}$-dependent random variables, estimating distributions, spatial data on integer lattices.

1. Introduction

It is well known that Efron's (1979) bootstrap, based on independent and identically distributed (i.i.d.) observations, provides good estimators in nonparametric statistical analysis. However, Efron's bootstrap fails if the observations are not independent (cf. Singh, 1981, Remark 2.1). For making the bootstrap suitable for dependent observations, various block resampling methods have been proposed. Instead of drawing one sample observation at a time, as in Efron's bootstrap, a block of sample observations is drawn where the size of the block should increase at some rate as the sample size gets larger. In the present paper, some block resampling methods for *spatially* dependent data are proposed. The specific use of blocks for such data was first studied by Hall (1985). Other contributions have been made by Hall (1988), Garcia-Soidan & Hall (1997), and Lahiri *et al.* (1999), among others. See also Chapter 5 in Politis *et al.* (1999), and the references therein.

Block resampling methods, like those suggested in the articles mentioned above, have been proposed to handle *stationary* spatial data. In this paper methods of estimating the distribution of sample means based on non-stationary spatial data are proposed.

Assume that we have some spatially indexed data, i.e. $\{X_{\boldsymbol{i}} : \boldsymbol{i} \in \mathcal{I}\}$, where $\mathcal{I}$ is a finite subset of the integer lattice $\mathbb{Z}^2$. Remote sensing data from satellites are, for example, of this form. For simplicity, the case when $\mathcal{I}$ is rectangular will be considered, but extensions to non-rectangular subsets of $\mathbb{Z}^2$, that possess some regularity, are possible. The kind of data we consider is of the following type: The values registered at different lattice cells are allowed to come from different distributions, and the dependence structure is allowed to differ over the lattice. We assume that all observed values are from distributions with the same expected value, or with expected values that can be decomposed additively into directional components. Furthermore, it will be assumed that observations separated by a certain distance are independent. The distribution of the sample mean under such conditions does not necessarily converge to a limit distribution. However, it will be shown that the sequence of distributions of sample means *weakly approaches* a sequence of normal distributions, for which Ekström (2001) provided consistent non-parametric variance estimators.

The concept of weakly approaching sequences was introduced by Belyaev (1995), and further developed by Belyaev & Sjöstedt-de Luna (2000). It is a natural generalization of weak convergence, without the need of a limit distribution.

A block resampling scheme is also suggested in the present paper, and it is shown that the proposed sequence of resampling estimators weakly approaches the sequence of distributions of sample means, in probability. The type of estimators considered in this paper has been studied earlier by Belyaev (1996), who considered triangular arrays of row-wise finitely dependent r.v.s, and proposed blockwise resampling schemes for estimating distributions of sums of r.v.s. Lemma 1-2 and Theorem 1-3 in the next section extend and modify results of Belyaev (1996) to spatial lattice data.

This paper is organized as follows: In the next section the estimators are introduced, and the obtained theoretical properties are presented. In Section 3, a simulation study is carried out in order to compare the different estimators. In Section 4 an approach, developed by the second author, for estimation of accuracy of discretely colored maps is considered. Proofs of lemmata are given in the Appendix.

We use the following notation: All sets are shown by calligraphic letters $\mathcal{A}$, and $|\mathcal{A}|$ is the number of elements in $\mathcal{A}$; r.v.s are denoted by capital letters; point in index position is used for sums, e.g. $X_{\cdot\cdot} = \sum_i \sum_j X_{ij}$; bold letters are used for vectors and lists; $\mathcal{L}(Z)$ is the distribution (probability measure) of a r.v. Z; $F_Z(\cdot)$ and $F_Z(\cdot \mid U)$ are the distribution function of Z and the regular conditional distribution function of Z given the r.v. U, respectively; $N(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2; $a \wedge b = \min\{a, b\}$; $a \vee b = \max\{a, b\}$; $a_n \asymp b_n$ is used for sequences $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$, for which there exist two constants $0 < c_1 < c_2 < \infty$ such that $c_1 \leq a_n/b_n \leq c_2$, $n = 1, 2, \ldots$. We use $\stackrel{P}{\longrightarrow}$ for convergence in probability and $\stackrel{w}{\longrightarrow}$ for weak convergence. Two sequences of r.v.s $\{U'_n\}_{n\geq 1}$ and $\{U''_n\}_{n\geq 1}$ have weakly approaching distribution laws $\{\mathcal{L}(U'_n)\}_{n\geq 1}$, $\{\mathcal{L}(U''_n)\}_{n\geq 1}$, if for any bounded continuous function $f(\cdot)$, $E[f(U'_n)] - E[f(U''_n)] \to 0$ as $n \to \infty$. Then we write $\mathcal{L}(U'_n) \stackrel{wa}{\longleftrightarrow} \mathcal{L}(U''_n)$, $n \to \infty$. Let $\{U'_n,\ Z_n\}_{n\geq 1}$ be a sequence of pairs of r.v.s and $\{\mathcal{L}(U'_n \mid Z_n)\}_{n\geq 1}$ be the sequence of regular conditional distribution laws of U'_n given Z_n. We write $\mathcal{L}(U'_n \mid Z_n) \stackrel{wa(P)}{\longleftrightarrow} \mathcal{L}(U''_n)$ if $E[f(U'_n) \mid Z_n] - E[f(U''_n)] \stackrel{P}{\longrightarrow} 0$, $n \to \infty$, for any bounded continuous function $f(\cdot)$.

2. Results

Assume we have spatially indexed data $\mathbb{X}_{\boldsymbol{n}} = \{X_{\boldsymbol{i}} : \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$, where $\mathcal{I}_{\boldsymbol{n}} = \{\boldsymbol{i} = \{i_1, i_2\} : i_1 = 1, ..., n_1, \text{ and } i_2 = 1, ..., n_2\}$, $\boldsymbol{n} = \{n_1, n_2\}$. Observations separated by a certain distance will be assumed to be independent, as formalized in the following definition.

Definition 1
The r.v.s $X_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, are said to be *spatially* $\boldsymbol{m}$*-dependent* if $X_{\boldsymbol{i}'}$ and $X_{\boldsymbol{i}''}$, $\boldsymbol{i}', \boldsymbol{i}'' \in \mathcal{I}_{\boldsymbol{n}}$, are independent whenever $|i_1' - i_1''| > m_1$ or $|i_2' - i_2''| > m_2$, $\boldsymbol{m} = \{m_1, m_2\}$.

Remark. The results in this paper hold for more general data than indicated, i.e., the results are valid for arrays $\mathbb{X}_{\boldsymbol{n}} = \{X_{\boldsymbol{i},\boldsymbol{n}} : \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$, $n_1, n_2 = 1, 2, ...$, of collections of r.v.s $X_{\boldsymbol{i},\boldsymbol{n}}$, such that for each $\boldsymbol{n}$, the r.v.s in $\mathbb{X}_{\boldsymbol{n}}$ are $\boldsymbol{m}$-dependent. To keep the notation manageable, we will write $X_{\boldsymbol{i}}$ rather than $X_{\boldsymbol{i},\boldsymbol{n}}$.

Define $m_{1,2} = m_1 m_2$ and $n_{1,2} = n_1 n_2$. Also, let $X.. = \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} X_{\boldsymbol{i}}$, $\bar{X}.. = X../n_{1,2}$, and $\gamma_{\boldsymbol{n}} = n_{1,2} Var[\bar{X}..]$. We are interested in the expected value $\bar{\mu}.. = E[\bar{X}..]$. $\bar{X}..$ is a point estimator of $\bar{\mu}..$, and in order to do inference, e.g., confidence intervals, we need to estimate the distribution law $\mathcal{L}(\sqrt{n_{1,2}}(\bar{X}.. - \bar{\mu}..))$. Below, we suggest different methods for estimating this distribution. We introduce the following assumptions:

AM: For all $\boldsymbol{n}$, $E[X_{\boldsymbol{i}}] = 0$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$.

AD$(\boldsymbol{m})$: For all $\boldsymbol{n}$, the r.v.s $X_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, are spatially $\boldsymbol{m}$-dependent.

AL(δ): For all $\boldsymbol{n}$, and some positive constants $\delta \leq 2$ and τ_δ, $E[|X_{\boldsymbol{i}}|^{2+\delta}] < \tau_\delta$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$.

By **AM** and **AD**$(\boldsymbol{m})$, we have

$$\begin{aligned}
\gamma_{\boldsymbol{n}} = \frac{1}{n_{1,2}} \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} \Bigg(& E[X_{\boldsymbol{i}}^2] + \sum_{h_1=1}^{m_1 \wedge (n_1 - i_1)} 2E[X_{\boldsymbol{i}} X_{i_1+h_1, i_2}] + \sum_{h_2=1}^{m_2 \wedge (n_2 - i_2)} 2E[X_{\boldsymbol{i}} X_{i_1, i_2+h_2}] \\
& + \sum_{h_1=1}^{m_1 \wedge (n_1 - i_1)} \sum_{h_2=1}^{m_2 \wedge (n_2 - i_2)} 2E[X_{\boldsymbol{i}} X_{i_1+h_1, i_2+h_2}] \\
& + \sum_{h_1=1}^{m_1 \wedge (n_1 - i_1)} \sum_{h_2=1}^{m_2 \wedge (i_2 - 1)} 2E[X_{\boldsymbol{i}} X_{i_1+h_1, i_2-h_2}] \Bigg). \qquad (1)
\end{aligned}$$

Theorem 1
If **AM**, **AD**$(\boldsymbol{m})$, *and* **AL**(δ) *are valid, then*

$$\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{..}) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}}), \ \ as\ n_1,n_2 \to \infty,$$

i.e., the sequences of distribution laws $\{\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{..})\}_{n_1,n_2\geq 1}$ *and* $\{N(0,\gamma_{\boldsymbol{n}})\}_{n_1,n_2\geq 1}$ *weakly approach each other as* $n_1,n_2 \to \infty$.

Remark. If in Theorem 1, $\gamma_{\boldsymbol{n}} \to \gamma > 0$, as $n_1,n_2 \to \infty$, then $\sqrt{n_{1,2}}\bar{X}_{..}$ converges weakly to $N(0,\gamma)$. This weak convergence also follows from more general results, e.g., from Theorem 6.1.2 in Lin & Lu (1996) on α-mixing random fields.

Let

$$\begin{aligned} \mathcal{S}_{2j-1,i} &= \{(j-1)k_i+(j-1)m_i+1,...,(jk_i+(j-1)m_i)\wedge n_i\}\,,\ i=1,2,\\ \mathcal{S}_{2j,i} &= \{jk_i+(j-1)m_i+1,...,(jk_i+jm_i)\wedge n_i\}\,,\ i=1,2, \end{aligned}$$

and define the following rectangular blocks of indices,

$$\begin{aligned} \mathcal{T}_{\boldsymbol{j}}^{(1)} &= \{\boldsymbol{i} : i_1 \in \mathcal{S}_{2j_1-1,1} \text{ and } i_2 \in \mathcal{S}_{2j_2-1,2}\}\,,\ \ \boldsymbol{j} \in \mathcal{J}^{(1)} = \{(1,1),...,(s_1,s_2)\},\\ \mathcal{T}_{\boldsymbol{j}}^{(2)} &= \{\boldsymbol{i} : i_1 \in \mathcal{S}_{2j_1-1,1} \text{ and } i_2 \in \mathcal{S}_{2j_2,2}\}\,,\ \ \boldsymbol{j} \in \mathcal{J}^{(2)} = \{(1,1),...,(s_1,t_2)\},\\ \mathcal{T}_{\boldsymbol{j}}^{(3)} &= \{\boldsymbol{i} : i_1 \in \mathcal{S}_{2j_1,1} \text{ and } i_2 \in \mathcal{S}_{2j_2-1,2}\}\,,\ \ \boldsymbol{j} \in \mathcal{J}^{(3)} = \{(1,1),...,(t_1,s_2)\},\\ \mathcal{T}_{\boldsymbol{j}}^{(4)} &= \{\boldsymbol{i} : i_1 \in \mathcal{S}_{2j_1,1} \text{ and } i_2 \in \mathcal{S}_{2j_2,2}\}\,,\ \ \boldsymbol{j} \in \mathcal{J}^{(4)} = \{(1,1),...,(t_1,t_2)\}, \end{aligned}$$

where $k_h > m_h$, and s_h, t_h, $h=1,2$, are the largest integers such that,

$$s_h \leq \frac{k_h+m_h+n_h-1}{k_h+m_h} \quad \text{and} \quad t_h \leq \frac{m_h+n_h-1}{k_h+m_h}, \quad s_h-t_h \leq 1. \tag{2}$$

We can write $\bar{X}_{..} = \sum_{l=1}^4 \bar{X}_{\boldsymbol{n}}^{(l)}$, where

$$\bar{X}_{\boldsymbol{n}}^{(l)} = \frac{1}{n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}^{(l)}} \sum_{\boldsymbol{i}\in\mathcal{T}_{\boldsymbol{j}}^{(l)}} X_{\boldsymbol{i}}.$$

Define $\gamma_{\boldsymbol{n}}^{(1)} = n_{1,2}Var[\bar{X}_{\boldsymbol{n}}^{(1)}]$.

Lemma 1
Under the assumptions of Theorem 1, and if $k_h = o(n_h)$ *as* $k_h,n_h \to \infty$, $h=1,2$,

$$(i) \quad \sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(l)} \overset{P}{\longrightarrow} 0,\ l=2,3,4, \quad and \quad (ii) \quad \gamma_{\boldsymbol{n}}^{(1)} - \gamma_{\boldsymbol{n}} \to 0,$$

as $n_1,n_2 \to \infty$.

Lemma 2 (Ekström, 2001)
Assume, for all $\boldsymbol{n}$ and $\boldsymbol{i}$, that $\alpha_{\boldsymbol{i}}=\alpha_{\boldsymbol{i}}(\boldsymbol{n})$ has an absolute value less than or equal to 1. Then, under the assumptions of Theorem 1, for some constant $\eta\geq 1$ and $b_h-a_h\geq m_h$, $h=1,2$,

$$(i)\qquad E\left[\left|\sum_{i_h=a_h+1}^{b_h}\alpha_{\boldsymbol{i}}X_{\boldsymbol{i}}\right|^{2+\delta}\right]\leq\tau_\delta\eta(8m_h(b_h-a_h))^{1+\delta/2},\quad h=1,2,$$

$$(ii)\qquad E\left[\left|\sum_{i_1=a_1+1}^{b_1}\sum_{i_2=a_2+1}^{b_2}\alpha_{\boldsymbol{i}}X_{\boldsymbol{i}}\right|^{2+\delta}\right]\leq\tau_\delta\eta(64m_{1,2}(b_1-a_1)(b_2-a_2))^{1+\delta/2}.$$

Proof of Theorem 1. The desired result is proved by Belyaev & Sjöstedt (1996) in the case when the r.v.s $X_{\boldsymbol{i}}, \boldsymbol{i}\in\mathcal{I}_{\boldsymbol{n}}$, are independent. That is, if **AM**, **AD**$(\mathbf{0})$, and **AL**(δ) are valid, then

$$\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot})\overset{wa}{\longleftrightarrow}N(0,\gamma_{\boldsymbol{n}}),\ \text{ as } n_1,n_2\to\infty. \tag{3}$$

Without loss of generality we assume that $m_1,m_2\geq 1$. By (1), the Cauchy-Schwarz, and the Lyapunov inequalities, respectively, one can verify that

$$\gamma_{\boldsymbol{n}}\leq\tau_\delta^{2/(2+\delta)}(1+2(m_1+m_2)+4m_{1,2}),\ \text{ for all } n_1,n_2, \tag{4}$$

and therefore the collection of r.v.s $\{\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot}\}_{n_1,n_2\geq 1}$ is tight. If the difference of elements of two random collections tends to zero in probability, and if one of the collections is tight, then the other collection is also tight and the collections weakly approach each other in distribution (Lemma 7, Belyaev & Sjöstedt-de Luna, 2000). Thus, by Lemma 1(i), it is enough to show that $\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(1)})\overset{wa}{\longleftrightarrow}N(0,\gamma_{\boldsymbol{n}})$, as $k_h/n_h\to 0$ and $k_h,n_h\to\infty$, $h=1,2$.

Consider the r.v.s,

$$X_{\boldsymbol{j}}^{(1)}=\sqrt{\frac{s_1s_2}{n_{1,2}}}\sum_{\boldsymbol{i}\in\mathcal{T}_{\boldsymbol{j}}^{(1)}}X_{\boldsymbol{i}},\ \ \boldsymbol{j}\in\mathcal{J}^{(1)}.$$

By their construction, they are independent and have expectation zero. Further, by Lemma 2(ii) and the upper bound (2) for s_h, for any $n_h>k_h\vee m_h$, $h=1,2$,

$$E\left[\left|X_{\boldsymbol{j}}^{(1)}\right|^{2+\delta}\right]\leq\tau_\delta\eta(64m_{1,2})^{1+\delta/2}\prod_{h=1}^{2}\left(\frac{s_hk_h}{n_h}\right)^{1+\delta/2}\leq\tau_\delta\eta(576m_{1,2})^{1+\delta/2}.$$

Hence, the r.v.s $X_{\boldsymbol{j}}^{(1)}$, $\boldsymbol{j}\in\mathcal{J}^{(1)}$, satisfy assumptions **AM**, **AD**$(\mathbf{0})$, and **AL**(δ) (with τ_δ replaced by $\tau_\delta\eta(576m_{1,2})^{1+\delta/2}$). Thus, by (3), with $X_{\boldsymbol{j}}^{(1)}$ instead of $X_{\boldsymbol{i}}$, s_h instead of n_h, and by noting that $\sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(1)} = \sum_{\boldsymbol{j}\in\mathcal{J}^{(1)}} X_{\boldsymbol{j}}^{(1)}/\sqrt{s_1 s_2}$, it follows that $\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(1)}) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}}^{(1)})$, as $s_1, s_2 \to \infty$.

Observe that inequality (4) implies that the collection of normal distributions $\{N(0,\gamma_{\boldsymbol{n}})\}_{\boldsymbol{n}}$ is tight as $n_1, n_2 \to \infty$. Two random collections, of which one is tight, weakly approach each other in distribution if and only if the difference of their characteristic functions tend to zero (Theorem 1, Belyaev & Sjöstedt-de Luna, 2000). From Lemma 1(ii), $\gamma_{\boldsymbol{n}}^{(1)} - \gamma_{\boldsymbol{n}} \to 0$, and so $N(0,\gamma_{\boldsymbol{n}}^{(1)}) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}})$, as $k_h/n_h \to 0$ and $k_h, n_h \to \infty$, $h = 1, 2$. Thus, we have

$$\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{..}) \overset{wa}{\longleftrightarrow} \mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(1)}) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}}^{(1)}) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}}),$$

as $k_h/n_h \to 0$ and $k_h, n_h \to \infty$, $h = 1, 2$, and this completes the proof. □

For making use of Theorem 1, we need to know or estimate the variance $\gamma_{\boldsymbol{n}}$. Define rectangular blocks of indices

$$\mathcal{B}_{\boldsymbol{j}} = \left\{ \begin{array}{ccc} \{j_1, j_2\} & \cdots & \{j_1+k_1-1, j_2\} \\ \vdots & \ddots & \vdots \\ \{j_1, j_2+k_2-1\} & \cdots & \{j_1+k_1-1, j_2+k_2-1\} \end{array} \right\}, \tag{5}$$

and let

$$X_{\boldsymbol{j}}^{\diamond} = \sum_{\boldsymbol{i}\in\mathcal{B}_{\boldsymbol{j}}} (X_{\boldsymbol{i}} - \bar{X}_{..})\, I_{\boldsymbol{i}}, \tag{6}$$

where $I_{\boldsymbol{i}}$ is equal to 1 if $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, and zero otherwise . In Ekström (2001) the following estimator of $\gamma_{\boldsymbol{n}}$ is proposed:

$$\hat{\gamma}_{\boldsymbol{n}} = \frac{1}{k_{1,2}n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}} \left(X_{\boldsymbol{j}}^{\diamond}\right)^2, \tag{7}$$

where $k_{1,2} = k_1 k_2 = |\mathcal{B}_{\boldsymbol{j}}|$ is the block size, and $\mathcal{J}_{\boldsymbol{n}} = \{\boldsymbol{j} = \{j_1, j_2\} : j_1 = 2-k_1, ..., n_1,$ and $j_2 = 2-k_2, ..., n_2\}$.

We make the following assumption on k_1 and k_2:

AK(δ): If $\delta = 2$, then $k_h = o(n_h)$ as $k_h, n_h \to \infty$, $h = 1, 2$. If $0 < \delta < 2$, then $(k_1/k_2)((k_1k_2/(n_1n_2))\log k_2)^{\delta}$, $(k_2/k_1)((k_1k_2/(n_1n_2))\log k_1)^{\delta}$, and $(k_h/n_h)\log k_h$, $h = 1, 2$, all tend to zero as $k_h, n_h \to \infty$, $h = 1, 2$.

Remark. If $n_1 \asymp n_2$, $k_1 \asymp k_2$, and $k_1 n_1^{-1} \log k_1 \to 0$, as $\min\{n_1, n_2, k_1, k_2\} \to \infty$, then **AK**($\delta$) holds for any $0 < \delta \le 2$.

Theorem 2 (Ekström, 2001)
If **AM**, **AD**($\boldsymbol{m}$), **AL**(δ), *and* **AK**(δ) *are valid, then*

$$\hat{\gamma}_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}} \overset{P}{\longrightarrow} 0, \;\; \text{as } n_1, n_2 \to \infty.$$

Remark. Alternatively, the estimators of variance suggested by Politis & Romano (1993) and Sherman (1996), which are consistent under the assumptions of Theorem 2, can be used. Further, if $\delta = 2$, then for some $c_1 > 0$, $E[(\hat{\gamma}_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}})^2] \leq c_1(k_{1,2}n_{1,2}^{-1} + k_1^{-2} + k_2^{-2})$. Thus, if $n_1 \asymp n_2$ and $k_h \asymp \sqrt{n_h}$, $h = 1, 2$, then $\sqrt{n_{1,2}}E[(\hat{\gamma}_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}})^2] = O(1)$. See Ekström (2001).

By Theorem 1 and 2, $N(0, \hat{\gamma}_{\boldsymbol{n}})$ can be used as an estimator of the distribution $\mathcal{L}(\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \bar{\mu}_{\cdot\cdot}))$. Alternatively, a different, possibly non-normal (except in the asymptotics), approximation can be used, as will be seen in the next theorem. Let $\mathfrak{B}_{\boldsymbol{k}}$ denote the collection of blocks $\{\mathcal{B}_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}\}$, and let $b_{1,2} = (n_1 + k_1 - 1)(n_2 + k_2 - 1)$ denote the total number of blocks in $\mathfrak{B}_{\boldsymbol{k}}$. Draw randomly $d_{1,2}$ blocks with replacement from $\mathfrak{B}_{\boldsymbol{k}}$ and let $M_{\boldsymbol{j}}^\star$ equal the number of times the block $\mathcal{B}_{\boldsymbol{j}}$ appears in the set $\mathfrak{B}_{\boldsymbol{k}}^\star$ of resampled blocks. Hence, $\{M_{\boldsymbol{j}}^\star, \boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}\}$ follow a multinomial distribution and $E^\star[M_{\boldsymbol{j}}^\star] = d_{1,2}b_{1,2}^{-1}$, $E^\star[(M_{\boldsymbol{j}}^\star)^2] = d_{1,2}b_{1,2}^{-1}(1 - b_{1,2}^{-1} + d_{1,2}b_{1,2}^{-1})$, and $E^\star[M_{\boldsymbol{j}'}^\star M_{\boldsymbol{j}''}^\star] = d_{1,2}b_{1,2}^{-2}(-1 + d_{1,2})$, $\boldsymbol{j}' \neq \boldsymbol{j}''$, where $E^\star$ denotes the expectation over the resampling distribution.

Let

$$\widetilde{X}_{\boldsymbol{n}}^\star = \frac{1}{n_{1,2}} \left(\frac{b_{1,2}}{k_{1,2}d_{1,2}} \right)^{1/2} \sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} M_{\boldsymbol{j}}^\star X_{\boldsymbol{j}}^\diamond.$$

The distribution of $\widetilde{X}_{\boldsymbol{n}}^\star$ depends both on the original randomness of $\mathbb{X}_{\boldsymbol{n}}$ and on randomness generated by resampling from the collection of blocks $\mathfrak{B}_{\boldsymbol{k}}$. We will consider the conditional distribution of $\widetilde{X}_{\boldsymbol{n}}^\star$ given the data $\mathbb{X}_{\boldsymbol{n}}$.

From (6) we have that

$$\sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} X_{\boldsymbol{j}}^\diamond = k_{1,2}X_{\cdot\cdot} - k_{1,2}n_{1,2}\bar{X}_{\cdot\cdot} = 0, \tag{8}$$

and thus $E^\star[\widetilde{X}_{\boldsymbol{n}}^\star | \mathbb{X}_{\boldsymbol{n}}] = 0$. Moreover, from (8) we see that

$$\sum_{\boldsymbol{j}' \in \mathcal{J}_{\boldsymbol{n}}} \sum_{\boldsymbol{j}'' \in \mathcal{J}_{\boldsymbol{n}}, \boldsymbol{j}'' \neq \boldsymbol{j}'} X_{\boldsymbol{j}'}^\diamond X_{\boldsymbol{j}''}^\diamond = - \sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} (X_{\boldsymbol{j}}^\diamond)^2,$$

and so

$$\begin{aligned} E^\star[(\sqrt{n_{1,2}}\widetilde{X}_{\boldsymbol{n}}^\star)^2 | \mathbb{X}_{\boldsymbol{n}}] &= \frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}} \sum_{\boldsymbol{j}' \in \mathcal{J}_{\boldsymbol{n}}} \sum_{\boldsymbol{j}'' \in \mathcal{J}_{\boldsymbol{n}}} E^\star[M_{\boldsymbol{j}'}^\star M_{\boldsymbol{j}''}^\star] X_{\boldsymbol{j}'}^\diamond X_{\boldsymbol{j}''}^\diamond \\ &= \frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}} \left(\frac{d_{1,2}}{b_{1,2}} \left(1 - \frac{1}{b_{1,2}} + \frac{d_{1,2}}{b_{1,2}} \right) - \frac{d_{1,2}}{b_{1,2}^2}(-1 + d_{1,2}) \right) \sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} (X_{\boldsymbol{j}}^\diamond)^2 = \hat{\gamma}_{\boldsymbol{n}}. \end{aligned}$$

Thus, the conditional variance of $\sqrt{n_{1,2}}\widetilde{X}_{\boldsymbol{n}}^\star$ given $\mathbb{X}_{\boldsymbol{n}}$ approaches $\gamma_{\boldsymbol{n}}$ as $n_1, n_2 \to \infty$.

Theorem 3
Under the assumptions of Theorem 2, and if $d_{1,2}\to\infty$ *as* $n_1,n_2\to\infty$,

$$\mathcal{L}(\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}|\mathbb{X}_{\boldsymbol{n}}) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot}),\ \ as\ n_1,n_2\to\infty,$$

i.e., the sequence of conditional distribution laws $\{\mathcal{L}(\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}|\mathbb{X}_{\boldsymbol{n}})\}_{n_1,n_2\geq 1}$ *weakly approaches* $\{\mathcal{L}(\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot})\}_{n_1,n_2\geq 1}$, *in probability, as* $n_1,n_2\to\infty$.

Remark. From Theorem 3 above, together with Lemma 9 in Belyaev & Sjöstedt-de Luna (2000), it follows that $\sup_x |F_{\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}}(x|\mathbb{X}_{\boldsymbol{n}}) - F_{\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot}}(x)| \overset{P}{\longrightarrow} 0$, as $n_1,n_2\to\infty$.

Let $\Psi(z)=e^z-1-z-z^2/2$.

Lemma 3
Under the assumptions of Theorem 3,

$$R_{\boldsymbol{n}} = \frac{d_{1,2}}{b_{1,2}}\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}\Psi\left(itX^\diamond_{\boldsymbol{j}}\left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2}\right) \overset{P}{\longrightarrow} 0,\ \ as\ n_1,n_2\to\infty.$$

Proof of Theorem 3. Without loss of generality, we assume that $m_1\wedge m_2\geq 1$. Note that $n_{1,2}(k_{1,2}d_{1,2}/b_{1,2})^{1/2}\widetilde{X}^\star_{\boldsymbol{n}}$ is a sum of $d_{1,2}$ i.i.d. r.v.s $X^{\diamond\star}_1,\ldots,X^{\diamond\star}_{d_{n_1n_2}}$, where $X^{\diamond\star}_1$ takes the values $X^\diamond_{\boldsymbol{j}}$, $\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}$, with probability $b^{-1}_{n_1n_2}$ on each value. By this fact and (8), the conditional characteristic function $\varphi^\star_{\boldsymbol{n}}(\cdot|\mathbb{X}_{\boldsymbol{n}})$ of $\widetilde{X}^\star_{\boldsymbol{n}}$ given the data $\mathbb{X}_{\boldsymbol{n}}$ can be written as

$$\begin{aligned}\varphi^\star_{\boldsymbol{n}}(t|\mathbb{X}_{\boldsymbol{n}}) &= E^\star\left[e^{it\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}}|\mathbb{X}_{\boldsymbol{n}}\right] = \left(\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}\frac{1}{b_{1,2}}\exp\left(itX^\diamond_{\boldsymbol{j}}\left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2}\right)\right)^{d_{1,2}}\\ &= \left(1-\frac{t^2}{2n_{1,2}k_{1,2}d_{1,2}}\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}(X^\diamond_{\boldsymbol{j}})^2+\frac{1}{b_{1,2}}\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}\Psi\left(itX^\diamond_{\boldsymbol{j}}\left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2}\right)\right)^{d_{1,2}}\\ &= \left(1-\frac{t^2\gamma_{\boldsymbol{n}}}{2d_{1,2}}+\frac{t^2}{2d_{1,2}}(\hat{\gamma}_{\boldsymbol{n}}-\gamma_{\boldsymbol{n}})+\frac{R_{\boldsymbol{n}}}{d_{1,2}}\right)^{d_{1,2}}.\end{aligned}$$

By Theorem 2 and Lemma 3, for any given $t\in\mathbb{R}$,

$$\varphi^\star_{\boldsymbol{n}}(t|\mathbb{X}_{\boldsymbol{n}})-\exp(-t^2\gamma_{\boldsymbol{n}}/2) \overset{P}{\longrightarrow} 0,\ \text{as}\ n_1,n_2\to\infty, \tag{9}$$

where $\exp(-t^2\gamma_{\boldsymbol{n}}/2)$ is the characteristic function of the normal distribution $N(0,\gamma_{\boldsymbol{n}})$. Inequality (4) implies that $\{N(0,\gamma_{\boldsymbol{n}})\}_{\boldsymbol{n}}$ is tight. Hence, by Theorem 2 in Belyaev & Sjöstedt-de Luna (2000), $\mathcal{L}(\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}|\mathbb{X}_{\boldsymbol{n}}) \overset{wa(P)}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}})$, as $n_1,n_2\to\infty$.

Denote the characteristic function of $\sqrt{n_{1,2}}\bar{X}_{\cdot\cdot}$ by $\varphi_{\boldsymbol{n}}(\cdot)$. By Theorem 1 in this paper and by Theorem 1 in Belyaev & Sjöstedt-de Luna (2000), $\varphi_{\boldsymbol{n}}(t)-\exp(-t^2\gamma_{\boldsymbol{n}}/2)\to 0$, as $n_1, n_2 \to \infty$, for any given $t \in \mathbb{R}$. From this result, together with (9), we get that $\varphi^\star_{\boldsymbol{n}}(t|\mathbb{X}_{\boldsymbol{n}}) - \varphi_{\boldsymbol{n}}(t) \xrightarrow{P} 0$, for any given t, as $n_1, n_2 \to \infty$. Hence, Theorem 2 in Belyaev & Sjöstedt-de Luna (2000) implies the desired result. $\square$

Similar results can be obtained under the following more general assumption on the first moment:

AM′: For all $\boldsymbol{n}$, $E[X_{\boldsymbol{i}}] = \mu$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, where μ may depend on $\boldsymbol{n}$.

Corollary 1
If **AM′**, **AD**$(\boldsymbol{m})$, *and* **AL**(δ) *are valid, then*

$$(i) \qquad \mathcal{L}(\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \mu)) \overset{wa}{\longleftrightarrow} N(0, \gamma_{\boldsymbol{n}}), \text{ as } n_1, n_2 \to \infty.$$

If, in addition, **AK**(δ) *is valid, then*

$$(ii) \qquad \hat{\gamma}_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}} \xrightarrow{P} 0, \text{ as } n_1, n_2 \to \infty,$$

and if $d_{1,2} \to \infty$ *as* $n_1, n_2 \to \infty$, *then*

$$(iii) \qquad \mathcal{L}(\sqrt{n_{1,2}}\widetilde{X}^\star_{\boldsymbol{n}}|\mathbb{X}_{\boldsymbol{n}}) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \mu)), \text{ as } n_1, n_2 \to \infty.$$

Proof. Note that the r.v.s $X^0_{\boldsymbol{i}} = X_{\boldsymbol{i}} - \mu$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, satisfy the assumptions of Theorem 1, and that $\bar{X}^0_{\cdot\cdot} = n^{-1}_{1,2} \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} X^0_{\boldsymbol{i}} = \bar{X}_{\cdot\cdot} - \mu$. This implies that $\mathcal{L}(\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \mu)) \overset{wa}{\longleftrightarrow} N(0, \gamma_{\boldsymbol{n}})$, as $n_1, n_2 \to \infty$, and thus (i) is proved. Since

$$X^{0\diamond}_{\boldsymbol{j}} = \sum_{\boldsymbol{i} \in \mathcal{B}_{\boldsymbol{j}}} (X^0_{\boldsymbol{i}} - \bar{X}^0_{\cdot\cdot}) I_{\boldsymbol{i}} = \sum_{\boldsymbol{i} \in \mathcal{B}_{\boldsymbol{j}}} (X_{\boldsymbol{i}} - \bar{X}_{\cdot\cdot}) I_{\boldsymbol{i}} = X^\diamond_{\boldsymbol{j}}$$

we see that neither $\hat{\gamma}_{\boldsymbol{n}}$ nor $\widetilde{X}^\star_{\boldsymbol{n}}$ depend on μ. Hence, Theorem 2 and 3 implies (ii) and (iii), respectively. $\square$

Below we consider a case when observed r.v.s do not have a constant mean.

AM″: For all $\boldsymbol{n}$, we have $Y_{\boldsymbol{i}} = \mu_{\boldsymbol{i}} + X_{\boldsymbol{i}}$, where $X_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, satisfy assumption **AM**. $\mu_{\boldsymbol{i}}$ decomposes additively into directional components, i.e., $\mu_{\boldsymbol{i}} = \mu + r_{i_2} + c_{i_1}$, where μ is the overall mean, r_{i_2}, $i_2 = 1, ..., n_2$, are the row effects, and c_{i_1}, $i_1 = 1, ..., n_1$, are the column effects. All effects, μ, r_{i_2}, $i_2 = 1, ..., n_2$, and c_{i_1}, $i_1 = 1, ..., n_1$, may depend on $\boldsymbol{n}$.

It should be noted that the row and column effects are defined as deviations from the overall mean so that $\sum_{i_2=1}^{n_2} r_{i_2} = 0$ and $\sum_{i_1=1}^{n_1} c_{i_1} = 0$.

Theorem 4
If $\mathbf{AM}''$, $\mathbf{AD}(\boldsymbol{m})$, *and* $\mathbf{AL}(\delta)$ *are valid, then*

$$\mathcal{L}(\sqrt{n_{1,2}}(\bar{Y}_{\cdot\cdot}-\mu)) \overset{wa}{\longleftrightarrow} N(0,\gamma_{\boldsymbol{n}}), \text{ as } n_1, n_2 \to \infty.$$

Proof. The result is an immediate consequence of Theorem 1 and the fact that $\bar{X}_{\cdot\cdot}=\bar{Y}_{\cdot\cdot}-\mu$. □

The ordinary-least-squares estimators of the effects are

$$\begin{aligned}
\hat{\mu} &= \bar{Y}_{\cdot\cdot} = \frac{Y_{\cdot\cdot}}{n_{1,2}} = \frac{1}{n_{1,2}}\sum_{\boldsymbol{i}\in\mathcal{I}_{\boldsymbol{n}}} Y_{\boldsymbol{i}}, \\
\hat{r}_{i_2} &= \frac{Y_{\cdot i_2}}{n_1} - \hat{\mu} = \frac{1}{n_1}\sum_{i_1=1}^{n_1} Y_{\boldsymbol{i}} - \hat{\mu}, \;\; i_2 = 1,...,n_2, \\
\hat{c}_{i_1} &= \frac{Y_{i_1\cdot}}{n_2} - \hat{\mu} = \frac{1}{n_2}\sum_{i_2=1}^{n_2} Y_{\boldsymbol{i}} - \hat{\mu}, \;\; i_1 = 1,...,n_1.
\end{aligned}$$

Thus, we estimate $\mu_{\boldsymbol{i}}$ with $\hat{\mu}_{\boldsymbol{i}}=\hat{\mu}+\hat{r}_{i_2}+\hat{c}_{i_1}$, and we can define residuals, $e_{\boldsymbol{i}}=Y_{\boldsymbol{i}}-\hat{\mu}_{\boldsymbol{i}}$, $\boldsymbol{i}\in\mathcal{I}_{\boldsymbol{n}}$.

Next we want to estimate the variance $\gamma_{\boldsymbol{n}}=n_{1,2}Var[\bar{Y}_{\cdot\cdot}]=n_{1,2}Var[\bar{X}_{\cdot\cdot}]$. It should be noticed that the r.v.s $X_{\boldsymbol{i}}$, $\boldsymbol{i}\in\mathcal{I}_{\boldsymbol{n}}$, are not observable, and that we therefore cannot use the "old" estimator $\hat{\gamma}_{\boldsymbol{n}}$ of the variance $\gamma_{\boldsymbol{n}}$. Further, we cannot replace the $X_{\boldsymbol{i}}$s with the $Y_{\boldsymbol{i}}$s in the formula for $\hat{\gamma}_{\boldsymbol{n}}$, since the varying mean values of the $Y_{\boldsymbol{i}}$s will then ruin the estimate of the variance $\gamma_{\boldsymbol{n}}$. We can, however, replace the $X_{\boldsymbol{i}}$s with the residuals, and so we obtain the following estimator of variance:

$$\hat{\gamma}'_{\boldsymbol{n}} = \frac{1}{k_{1,2}n_{1,2}}\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}\left(Y^{\diamond}_{\boldsymbol{j}}\right)^2,$$

where $Y^{\diamond}_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{B}_{\boldsymbol{j}}} e_{\boldsymbol{i}} I_{\boldsymbol{i}}$.

Theorem 5 (Ekström, 2001)
If $\mathbf{AM}''$, $\mathbf{AD}(\boldsymbol{m})$, $\mathbf{AL}(\delta)$, *and* $\mathbf{AK}(\delta)$ *are valid, then*

$$\hat{\gamma}'_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}} \overset{P}{\longrightarrow} 0, \text{ as } n_1, n_2 \to \infty.$$

Remark. If $\delta=2$, then for some $c_2>0$, $E(\hat{\gamma}'_{\boldsymbol{n}}-\gamma_{\boldsymbol{n}})^2 \le c_2(k_{1,2}n_{1,2}^{-1}+k_1^{-2}+k_2^{-2}+k_1^2n_1^{-2}+k_2^2n_2^{-2})$. Thus, if $n_1 \asymp n_2$ and $k_h \asymp \sqrt{n_h}$, $h=1,2$, then $\sqrt{n_{1,2}}E(\hat{\gamma}'_{\boldsymbol{n}}-\gamma_{\boldsymbol{n}})^2 = O(1)$. See Ekström (2001).

By Theorems 4 and 5, $N(0, \hat{\gamma}'_{\boldsymbol{n}})$ can be used as an estimator of the distribution of $\sqrt{n_{1,2}}(\bar{Y}_{\cdot\cdot} - \mu)$. Alternatively, the following technique can be used. Let $\mathbb{Y}_{\boldsymbol{n}} = \{Y_{\boldsymbol{i}} : \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$, and define

$$\widetilde{Y}^{\star}_{\boldsymbol{n}} = \frac{1}{n_{1,2}} \left(\frac{b_{1,2}}{k_{1,2} d_{1,2}} \right)^{1/2} \sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} M^{\star}_{\boldsymbol{j}} Y^{\diamond}_{\boldsymbol{j}}.$$

Theorem 6
Under the assumptions of Theorem 5, and if $d_{1,2} \to \infty$ as $n_1, n_2 \to \infty$,

$$\mathcal{L}(\sqrt{n_{1,2}}\widetilde{Y}^{\star}_{\boldsymbol{n}} | \mathbb{Y}_{\boldsymbol{n}}) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(\sqrt{n_{1,2}}(\bar{Y}_{\cdot\cdot} - \mu)), \quad \textit{as } n_1, n_2 \to \infty.$$

Lemma 4
Under the assumptions of Theorem 6,

$$R'_{\boldsymbol{n}} = \frac{d_{1,2}}{b_{1,2}} \sum_{\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}} \Psi \left(it Y^{\diamond}_{\boldsymbol{j}} \left(\frac{b_{1,2}}{n_{1,2} k_{1,2} d_{1,2}} \right)^{1/2} \right) \xrightarrow{P} 0, \quad \textit{as } n_1, n_2 \to \infty.$$

Proof of Theorem 6. The proof is almost identical to the proof of Theorem 3. Only some changes of notation are needed, and the references to Theorem 1 and 2, and Lemma 3, should be changed to Theorem 4 and 5, and Lemma 4, respectively. □

Remark. It is easily seen that the results in this paper hold also for rectangular index sets $\mathcal{I}$ expanding in all directions, i.e. for $\mathcal{I} = \{\boldsymbol{i} : i_1 = -n_1, ..., n_1 \text{ and } i_2 = -n_2, ..., n_2\}$. Moreover, the assumption on the index set to be of rectangular shape can be relaxed by using "subshapes", as described in, for example, Sherman (1996).

3. A simulation study

In this Monte Carlo study, non-stationary spatially $\boldsymbol{m}$-dependent data $X_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$, are generated, where each $X_{\boldsymbol{i}}$ is a weighted average of independent and skewly distributed r.v.s such that $X_{\boldsymbol{i}}$ has a small variance if both i_1 and i_2 are small, whereas the variance of $X_{\boldsymbol{i}}$ is large when i_1 and i_2 are large. To be more specific, let $m_h = 2l_h$ for some integer $l_h \geq 0$, $h = 1, 2$, and define weights $w(\boldsymbol{i}) = v_{\boldsymbol{i}} / \sum_{j_1=-l_1}^{l_1} \sum_{j_2=-l_2}^{l_2} v_{\boldsymbol{j}}$, where $v_{\boldsymbol{i}} = ((1 + |i_1|)(1 + |i_2|))^{-1}$, $i_1 = -l_1, ..., l_1$, $i_2 = -l_2, ..., l_2$. Define

$$X_{\boldsymbol{i}} = \sum_{j_1 = i_1 - l_1}^{i_1 + l_1} \sum_{j_2 = i_2 - l_2}^{i_2 + l_2} w(\{|i_1 - j_1|, |i_2 - j_2|\}) \left(Z_{\boldsymbol{j}} - E[Z_{\boldsymbol{j}}] \right),$$

where the r.v.s Z_j are independent and log-normal with parameters $\theta = E(\log Z_j) = 0$, and

$$\sigma_j = \sqrt{Var(\log Z_j)} = \frac{1}{2}\sum_{h=1}^{2}\frac{j_h + l_h}{n_h + 2l_h + 1}, \quad \text{for all } \boldsymbol{j}.$$

For judging the performance of an estimate $\hat{F}_n$, of the distribution F_n of $\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \mu)$, we calculate the maximum absolute difference $\sup_x |\hat{F}_n(x) - F_n(x)|$, where F_n will be represented by the empirical distribution of 1 million replicates of $\sqrt{n_{1,2}}(\bar{X}_{\cdot\cdot} - \mu)$. Three different estimators of F_n will be considered:

NA: Normal approximation with the variance estimated by $\hat{\gamma}_n$;

R1: Resampling estimator $\sqrt{n_{1,2}}\widetilde{X}^{\star}_n$ with $d_{1,2} = b_{1,2}$ (this choice of $d_{1,2}$ corresponds to the number of resampled blocks used by Belyaev (1996) & Sjöstedt (2000) in their resampling schemes for sequences of m-dependent r.v.s); and

R2: Resampling estimator $\sqrt{n_{1,2}}\widetilde{X}^{\star}_n$ with $d_{1,2} = \lceil n_{1,2}/k_{1,2}\rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

In our study, 250 samples of $\mathbb{X}_n$ are generated for each choice of rectangular shapes of the index and blocks sets, $N = n_1 \times n_2$, $K = k_1 \times k_2$, and $\boldsymbol{m} = \{m_1, m_2\}$. The maximum absolute differences for each of the three estimators of F_n are calculated. In Table 1-7, the mean ($Mean$) and the standard deviation ($StDev$) of the maximum absolute differences are presented for each estimator.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
5×5	0.059	0.031	0.078	0.029	0.079	0.030
10×10	0.065	0.040	0.085	0.037	0.089	0.038
15×15	0.079	0.048	0.097	0.046	0.108	0.043

Table 1. $N = 25 \times 25$ *and* $\boldsymbol{m} = \{1, 1\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
5×5	0.085	0.041	0.102	0.039	0.104	0.037
10×10	0.080	0.047	0.098	0.044	0.106	0.043
15×15	0.084	0.055	0.102	0.052	0.113	0.050

Table 2. $N = 25 \times 25$ *and* $\boldsymbol{m} = \{2, 2\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
5×5	0.045	0.022	0.067	0.021	0.069	0.021
10×10	0.037	0.025	0.061	0.024	0.063	0.023
15×15	0.046	0.032	0.070	0.029	0.071	0.031

Table 3. $N = 50 \times 50$ *and* $\boldsymbol{m} = \{1, 1\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
5×5	0.078	0.025	0.098	0.025	0.097	0.025
10×10	0.049	0.029	0.070	0.027	0.074	0.027
15×15	0.053	0.034	0.075	0.031	0.079	0.032

Table 4. $N = 50 \times 50$ *and* $\boldsymbol{m} = \{2, 2\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
15×15	0.017	0.009	0.045	0.014	0.045	0.014
20×20	0.016	0.010	0.045	0.012	0.045	0.014
25×25	0.018	0.011	0.045	0.013	0.046	0.015
30×30	0.017	0.012	0.045	0.014	0.047	0.014

Table 5. $N = 250 \times 250$ *and* $\boldsymbol{m} = \{1, 1\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
15×15	0.025	0.012	0.050	0.015	0.052	0.014
20×20	0.021	0.012	0.046	0.013	0.048	0.016
25×25	0.020	0.013	0.047	0.015	0.047	0.014
30×30	0.020	0.013	0.048	0.016	0.049	0.016

Table 6. $N = 250 \times 250$ *and* $\boldsymbol{m} = \{2, 2\}$.

	NA		R1		R2	
Block size (K)	*Mean*	*StDev*	*Mean*	*StDev*	*Mean*	*StDev*
15×15	0.033	0.013	0.057	0.016	0.057	0.014
20×20	0.026	0.013	0.051	0.015	0.052	0.015
25×25	0.023	0.014	0.049	0.015	0.051	0.016
30×30	0.024	0.015	0.051	0.017	0.051	0.016

Table 7. $N = 250 \times 250$ *and* $\boldsymbol{m} = \{3, 3\}$.

In Table 1-7 we see that the normal approximation performs the best, even for rather small values of N. Further, for the resampling estimator $\sqrt{n_{1,2}}\widetilde{X}_{\boldsymbol{n}}^{\star}$ there is no real gain in choosing the number $d_{1,2}$ of resampled blocks as large as $b_{1,2}$; $d_{1,2} = \lceil n_{1,2}/k_{1,2} \rceil + 1$ seems quite enough. To state guidelines on how to choose an optimal block size in a given situation is a difficult task under the assumptions given in this paper, and will not be discussed here. It is clear, however, that the block size should increase with n_1 and n_2. Also, the optimal block size depends on the strength of dependence, as seen in the tables above.

The estimators $N(0, \hat{\gamma}_n')$, $\mathcal{L}(\sqrt{n_{1,2}}\widetilde{Y}_{\boldsymbol{n}}^{\star})$ with $d_{1,2} = b_{1,2}$, and $\mathcal{L}(\sqrt{n_{1,2}}\widetilde{Y}_{\boldsymbol{n}}^{\star})$ with $d_{1,2} = \lceil n_{1,2}/k_{1,2} \rceil + 1$, gave less satisfactory results in our simulations, and we do not recommend these estimators unless n_1 and n_2 are larger than in the simulations in this paper. An alternative to these estimators will be presented in a forthcoming paper.

4. An application to assessing characteristics of accuracy of discretely colored digital maps

The methods considered in this paper can be applied to assessing characteristics of accuracy of discretely colored maps created by using remote sensing data (Belyaev 2000a, b). For example, different types of landscapes can be shown in maps by using different colors. Let $\mathcal{K} = \{1, 2, ..., q\}$ be the number of colors used. A digital discretely colored map is a collection of small colored squares called pixels. We consider the lattice $\mathbb{Z}^2 = \{\boldsymbol{i} = \{i_1, i_2\} : i_1, i_2 = 0, 1, 2, ...\}$, and the $\boldsymbol{i}$-pixel is the square with the vertices $\{i_1, i_2\}, \{i_1+1, i_2\}, \{i_1+1, i_2+1\}, \{i_1, i_2+1\}$, whose color is coded by a digit $c_{\boldsymbol{i}} \in \mathcal{K}$. We denote pixels as $\{\boldsymbol{i}, c_{\boldsymbol{i}}\},\ \boldsymbol{i} \in \mathbb{Z}^2$. The same area can be shown by maps having different number of pixels. We consider a rectangular digital map $\mathfrak{M}_{\boldsymbol{n}} = \{\{\boldsymbol{i}, c_{\boldsymbol{i}}\} : \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$ where as above $\mathcal{I}_{\boldsymbol{n}} = \{\boldsymbol{i} = \{i_1, i_2\} :\ i_1 = 1, ..., n_1,\ i_2 = 1, ..., n_2\}$. Suppose that all pixels are correctly colored, i.e. $\mathfrak{M}_{\boldsymbol{n}}$ is the true map with $n_{1,2} = n_1 n_2$ pixels. Usually it is impossible or too expensive to create the true maps. Instead remotely sensed data $\mathbb{S}$, e.g. data registered by satellite sensors, are used to obtain approximations to the true maps. Suppose that some classification based on $\mathbb{S}$ is used in selecting colors of pixels. Denote by $c^\bullet_{\boldsymbol{i}}$ the (number of the) color obtained after classifying a part of the data $\boldsymbol{s}_{\boldsymbol{i}} \in \mathbb{S}$ related to the $\boldsymbol{i}$-pixel. Assume that the cross-classification probabilities $p_{kl}(\boldsymbol{s}) = P(c^\bullet_{\boldsymbol{i}} = l \mid c_{\boldsymbol{i}} = k,\ \boldsymbol{s}_{\boldsymbol{i}} = \boldsymbol{s})$ are known. Here, $p_{kl}(\boldsymbol{s})$ is the conditional probability to select color $c^\bullet_{\boldsymbol{i}} = l$ given the true color $c_{\boldsymbol{i}} = k$ and the related remotely sensed data $\boldsymbol{s} = \boldsymbol{s}_{\boldsymbol{i}}$. For sake of simplicity we assume that $l \in \mathcal{K}$ and square matrices $\mathbb{P}(\boldsymbol{s}) = (p_{kl}(\boldsymbol{s})),\ \boldsymbol{s} \in \mathbb{S}$, have full rank. Then there exist inverse matrices $\mathbb{P}^{-1}(\boldsymbol{s}) = (p^{kl}(\boldsymbol{s})),\ \boldsymbol{s} \in \mathbb{S}$. The numbers $N^\bullet_{kl}$ of pixels which have the true color k and have been classified as having the color l, are important characteristics of accuracy of the created map $\mathfrak{M}^\bullet_{\boldsymbol{n}} = \{\{\boldsymbol{i}, c^\bullet_{\boldsymbol{i}}\} :\ \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$. By using the indicator functions we can write $N^\bullet_{kl}$ as follows:

$$N^\bullet_{kl} = \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} I(c_{\boldsymbol{i}} = k) I(c^\bullet_{\boldsymbol{i}} = l). \tag{10}$$

Its mathematical expectation is

$$E[N^\bullet_{kl}] = \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} I(c_{\boldsymbol{i}} = k) p_{kl}(\boldsymbol{s}_{\boldsymbol{i}}). \tag{11}$$

Both values $N^\bullet_{kl}$ and $E[N^\bullet_{kl}]$ are not feasible. However, there exist the unbiased estimators $E[N^\bullet_{kl}]$

$$\hat{N}^\bullet_{kl} = \sum_{\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}} \hat{I}(c_{\boldsymbol{i}} = k) p_{kl}(\boldsymbol{s}_{\boldsymbol{i}}), \tag{12}$$

where

$$\hat{I}(c_{\boldsymbol{i}} = k) = \sum_{l \in \mathcal{K}} p^{lk}(\boldsymbol{s}) I(c^\bullet_{\boldsymbol{i}} = l). \tag{13}$$

We will consider digital maps with very large numbers of pixels $\boldsymbol{n}$. For simplicity, we will consider the asymptotic behavior as $n_1, n_2 \to \infty$. We can consistently estimate variances and distributions of normed deviations $(\hat{N}^\bullet_{kl} - E[N^\bullet_{kl}])/\sqrt{n_{1,2}},\ k,l \in \mathcal{K}$. Let $g_{1,2}$ be the number of all boundary pixels, i.e., the pixels whose colors differ from the colors of some of their neighboring pixels. We need the following assumption:

AB: $\quad g_{1,2}/(n_1 \wedge n_2) \leq c < \infty \quad$ as $\quad n_1, n_2 \to \infty.$

In words this assumption means that the number of pixels in $\mathfrak{M}_{\boldsymbol{n}}$, constituting the boundaries dividing subsets of pixels having the same true colors, is comparable with the minimal diameter of $\mathcal{I}_{\boldsymbol{n}}$. Heuristically, this means that the boundaries dividing subsets of pixels having the same colors in the true map $\mathfrak{M}_{\boldsymbol{n}}$ can be well approximated by smooth curves.

The r.v.s $X_{\boldsymbol{i}}$ considered in Section 2 will be defined here by the following relation:

$$X_{\boldsymbol{i}} = \hat{I}(c_{\boldsymbol{i}} = k)p_{kl}(\boldsymbol{s}) - I(c_{\boldsymbol{i}} = k)p_{kl}(\boldsymbol{s}), \quad \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}. \tag{14}$$

We will use the same collection $\mathfrak{B}_{\boldsymbol{k}}$ of blocks $\mathcal{B}_{\boldsymbol{j}}$ defined by (5) and suppose that the results of classification $\{c^\bullet_{\boldsymbol{i}} :\ \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$ are $\boldsymbol{m}$-dependent, $\boldsymbol{m} = \{m_1, m_2\}$. Then $\mathbf{AD}(\boldsymbol{m})$ holds for $\mathbb{X}_{\boldsymbol{n}}$. We also introduce the following assumptions:

AC: $\quad \sup_{\boldsymbol{s}\in\mathbb{S}} \max_{k,l} |p^{kl}(\boldsymbol{s})| \leq c < \infty,$

AK′: $\quad k_1 \asymp k_2,\ k_1/\sqrt{n_1} \to 0,$ and $k_2/\sqrt{n_2} \to 0, \quad$ as $\quad n_1, n_2 \to \infty.$

From **AC** it follows that all $|X_{\boldsymbol{i}}|$ are uniformly bounded and $\mathbf{AL}(\delta)$ holds for any $\delta > 0$. If $\mathbf{AK}'$ holds then $\mathbf{AK(2)}$ also holds. Hence, all results stated in Section 2 are valid for r.v.s $X_{\boldsymbol{i}},\ \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}$ defined by (14). But we can not directly use them in assessing distribution of deviations

$$\frac{\hat{N}^\bullet_{kl} - E[N^\bullet_{kl}]}{\sqrt{n_{1,2}}} = \frac{1}{\sqrt{n_{1,2}}} \sum_{\boldsymbol{i}\in\mathcal{I}_{\boldsymbol{n}}} X_{\boldsymbol{i}} \tag{15}$$

because the values of indicators $I(c_{\boldsymbol{i}} = k)$ are not observable. Henceforth, in order to be brief, we suppose that m_1, m_2, k_1, k_2 are even.

We can solve the problem of assessing variance and the distribution function of deviations (15) by using instead of $X^\circ_{\boldsymbol{j}}$ the following r.v.s:

$$X^\circ_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{B}_{\boldsymbol{j}}} (-1)^{I(i_1 > j_1 + k_1/2) + I(i_2 > j_2 + k_2/2)} \hat{I}(c_{\boldsymbol{i}} = k)p_{kl}(\boldsymbol{s}).$$

Let

$$X^\bullet_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{B}_{\boldsymbol{j}}} (-1)^{I(i_1 > j_1 + k_1/2) + I(i_2 > j_2 + k_2/2)} (X_{\boldsymbol{i}} - \bar{X}..)I_{\boldsymbol{i}},$$

and

$$\hat{\gamma}^\circ_{\boldsymbol{n}} = \frac{1}{k_{1,2}n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}} (X^\circ_{\boldsymbol{j}})^2.$$

Theorem 7
Suppose that the random pixels' colors $\mathcal{C}^\bullet_{\boldsymbol{n}} = \{c^\bullet_{\boldsymbol{i}},\ \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}}\}$ in the created map $\mathfrak{M}^\bullet_{\boldsymbol{n}}$ are $\boldsymbol{m}$-dependent and assumptions **AB**, **AC** *and* **AK′** *hold. Then*

$$\hat{\gamma}^\circ_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}} \xrightarrow{P} 0, \quad \textit{as} \quad n_1, n_2 \to \infty. \tag{16}$$

The proof of (16) is rather long. Here we give a short sketch of it. In the proof we use the equality $X^\circ_{\boldsymbol{j}} = X^\bullet_{\boldsymbol{j}}$ if $\mathcal{B}_{\boldsymbol{j}}$ is inside of $\mathcal{I}_{\boldsymbol{n}}$ and it does not contain boundary pixels in $\mathfrak{M}_{\boldsymbol{n}}$. Assumption **AB** implies that the share of such blocks tends to one as $n_1, n_2 \to \infty$. Let $\mathcal{A}^{(1)}(j,k,m) = \{i :\ j \le i \le j + (k-m)/2\}$ and $\mathcal{A}^{(2)}(j,k,m) = \{i :\ j + (k+m)/2 < i \le j + k\}$ and let $\mathcal{B}^{(s,t)}_{\boldsymbol{j}}\{\boldsymbol{i} :\ i_1 \in \mathcal{A}^{(s)}(j_1,k_1,m_1),\ i_2 \in \mathcal{A}^{(t)}(j_2,k_2,m_2)\},\ s,t = 1,2$. We define the following r.v.s:

$$X^{(s,t)}_{\boldsymbol{j}} = (-1)^{s+t} \sum_{\boldsymbol{i}\in\mathcal{B}^{(s,t)}_{\boldsymbol{j}}} (X_{\boldsymbol{i}} - \bar{X}_{\cdot\cdot}) I_{\boldsymbol{i}}.$$

Further we prove that

$$\hat{\gamma}^\circ_{\boldsymbol{n}} = \frac{1}{k_{1,2}n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}} \sum_{s,t=1,2} (X^{(s,t)}_{\boldsymbol{j}})^2 + R^\circ_{\boldsymbol{n}},$$

where $R^\circ_{\boldsymbol{n}} \xrightarrow{P} 0$ as $n_1, n_2 \to \infty$. We then use Theorem 2 in Ekström (2001) for each sum $(k'_{1,2}n_{1,2})^{-1}\sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}}(X^{(s,t)}_{\boldsymbol{j}})^2$ with $k'_{1,2} = (k_1/2)(k_2/2)$, and we obtain relation (16). The random sampling of $d_{1,2}$ blocks $\mathcal{B}_{\boldsymbol{j}} \in \mathfrak{B}_{\boldsymbol{k}},\ \boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}$, can also be applied here. Let

$$\widetilde{X}^{\circ\star}_{\boldsymbol{n}} = \frac{1}{n_{1,2}} \left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2} \sum_{\boldsymbol{j}\in\mathcal{J}_{\boldsymbol{n}}} M^\star_{\boldsymbol{j}} X^\circ_{\boldsymbol{j}}.$$

Theorem 8
Under the assumptions of Theorem 7,

$$\mathcal{L}(\widetilde{X}^{\circ\star}_{\boldsymbol{n}} \mid \mathcal{C}^\bullet_{\boldsymbol{n}}) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(\frac{\hat{N}^\bullet_{kl} - E[N_{kl}]}{\sqrt{n_{1,2}}}\right), \quad \textit{as} \quad n_1, n_2 \to \infty. \tag{17}$$

The proof of (17) can be obtained by using the same method as in Section 2. The detailed proof of Theorem 7 and 8 will be given in a separate paper with several numerical examples. Presently we do not know how large n_1, n_2 should be so that it would be possible to apply the suggested asymptotic approach.

5. Appendix

Two useful inequalities:

Inequality A: For any positive numbers $z_1, ..., z_r$ and $\lambda \geq 1$ we have, from the Jensen inequality,

$$(z_1 + ... + z_r)^\lambda \leq r^{\lambda-1}(z_1^\lambda + ... + z_r^\lambda).$$

Inequality B: For any independent random variables $Z_1, ..., Z_r$ and $\lambda \geq 1$, with $E[Z_h] = 0$ and $E[|Z_h|^\lambda] < \infty$, $h = 1, ..., r$, it holds that

$$E[|Z_1 + ... + Z_r|^\lambda] \leq \eta r^{(\lambda/2-1)\vee 0}(E[|Z_1|^\lambda] + ... + E[|Z_r|^\lambda]),$$

where $1 \leq \eta = \eta(\lambda) < \infty$ is a constant, and $\eta \leq 2$ if $\lambda \leq 2$ (Petrov 1995, p. 82-83).

Proof of Lemma 1(i). Note that $X_{\boldsymbol{i}'}$ and $X_{\boldsymbol{i}''}$ are independent when $\boldsymbol{i}'$ and $\boldsymbol{i}''$ belong to two different blocks $\mathcal{T}_{\boldsymbol{j}'}^{(2)}$ and $\mathcal{T}_{\boldsymbol{j}''}^{(2)}$. Thus,

$$E[(\bar{X}_{\boldsymbol{n}}^{(2)})^2] = \frac{1}{n_{1,2}^2} \sum_{\boldsymbol{j} \in \mathcal{J}^{(2)}} E\left[\left(\sum_{\boldsymbol{i} \in \mathcal{T}_{\boldsymbol{j}}^{(2)}} X_{\boldsymbol{i}}\right)^2\right].$$

Define

$$\begin{aligned} \mathcal{S}_{2j-1,i,h} &= \{(j-1)k_i + (j-1)m_i + (h-1)m_i + 1, ..., \\ &\qquad ((j-1)k_i + (j-1)m_i + hm_i) \wedge (jk_i + (j-1)m_i) \wedge n_i\}. \end{aligned}$$

Further, define

$$\mathcal{T}_{\boldsymbol{j},h}^{(2)} = \{\boldsymbol{i} : i_1 \in \mathcal{S}_{2j_1-1,1,h} \text{ and } i_2 \in \mathcal{S}_{2j_2,2}\}, \boldsymbol{j} \in \mathcal{J}^{(2)} \text{ and } h = 1, ..., l,$$

where $l \leq (k_1 + m_1 - 1)/m_1$, and

$$\mathcal{U}_{\boldsymbol{j},g}^{(2)} = \left\{ h : \mathcal{T}_{\boldsymbol{j},h}^{(2)} \in \bigcup_{f: g+2(f-1) \leq l,\, f \geq 1} \mathcal{T}_{\boldsymbol{j},g+2(f-1)}^{(2)} \right\}.$$

By Inequality A with $\lambda = 2$ and $r = 2$, Inequality B with $\lambda = 2$ and $r = |\mathcal{U}_{\boldsymbol{j},g}^{(2)}|$, and Inequality A with $\lambda = 2$ and $r \leq M$, respectively,

$$n_{1,2}E[(\bar{X}_{\boldsymbol{n}}^{(2)})^2] = \frac{1}{n_{1,2}} \sum_{\boldsymbol{j} \in \mathcal{J}^{(2)}} E\left[\left(\sum_{g=1}^{2} \sum_{h \in \mathcal{U}_{\boldsymbol{j},g}^{(2)}} \sum_{\boldsymbol{i} \in \mathcal{T}_{\boldsymbol{j},h}^{(2)}} X_{\boldsymbol{i}}\right)^2\right]$$

$$\leq \frac{2}{n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}^{(2)}} \sum_{g=1}^{2} E\left[\left(\sum_{h\in\mathcal{U}_{\boldsymbol{j},g}^{(2)}} \sum_{\boldsymbol{i}\in\mathcal{T}_{\boldsymbol{j},h}^{(2)}} X_{\boldsymbol{i}}\right)^2\right]$$

$$\leq \frac{4}{n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}^{(2)}} \sum_{g=1}^{2} \sum_{h\in\mathcal{U}_{\boldsymbol{j},g}^{(2)}} E\left[\left(\sum_{\boldsymbol{i}\in\mathcal{T}_{\boldsymbol{j},h}^{(2)}} X_{\boldsymbol{i}}\right)^2\right]$$

$$\leq \frac{4m_{1,2}}{n_{1,2}} \sum_{\boldsymbol{j}\in\mathcal{J}^{(2)}} \sum_{g=1}^{2} \sum_{h\in\mathcal{U}_{\boldsymbol{j},g}^{(2)}} \sum_{\boldsymbol{i}\in\mathcal{T}_{\boldsymbol{j},h}^{(2)}} E[X_{\boldsymbol{i}}^2]$$

$$\leq \frac{4m_{1,2}(k_1+m_1+n_1)(m_2+n_2)k_1m_2\tau_\delta^{2/(2+\delta)}}{n_{1,2}(k_1+m_1)(k_2+m_2)}, \tag{18}$$

where the last inequality follows from (2) and the inequality $|\mathcal{T}_{\boldsymbol{j},h}^{(2)}| \leq k_1 m_2$. Likewise,

$$n_{1,2}E[(\bar{X}_{\boldsymbol{n}}^{(3)})^2] \leq \frac{4m_{1,2}(m_1+n_1)(k_2+m_2+n_2)k_2m_1\tau_\delta^{2/(2+\delta)}}{n_{1,2}(k_1+m_1)(k_2+m_2)}, \tag{19}$$

$$n_{1,2}E[(\bar{X}_{\boldsymbol{n}}^{(4)})^2] \leq \frac{4m_{1,2}^2(m_1+n_1)(m_2+n_2)\tau_\delta^{2/(2+\delta)}}{n_{1,2}(k_1+m_1)(k_2+m_2)}. \tag{20}$$

By the assumptions on k_h and n_h, $h=1,2$, we see that $n_{1,2}E[(\bar{X}^{(l)})^2] \to 0$, $l=2,3,4$, and so $\sqrt{n_{1,2}}\bar{X}_{\boldsymbol{n}}^{(l)} \xrightarrow{P} 0$, $l=2,3,4$, as $k_h/n_h \to 0$ and $k_h, n_h \to \infty$, $h=1,2$. □

Proof of Lemma 1(ii). By the Cauchy-Schwarz inequality, and inequalities (4) and (18)-(20),

$$|\gamma_{\boldsymbol{n}} - \gamma_{\boldsymbol{n}}^{(1)}| = n_{1,2}\left|E\left[\left(\sum_{l=1}^{4}\bar{X}_{\boldsymbol{n}}^{(l)}\right)^2\right] - E[(\bar{X}_{\boldsymbol{n}}^{(1)})^2]\right|$$

$$= n_{1,2}\left|\sum_{l=2}^{4}E[(\bar{X}_{\boldsymbol{n}}^{(l)})^2] - 2\sum_{l=1}^{4}\sum_{j=l+1}^{4}E[\bar{X}_{\boldsymbol{n}}^{(l)}\bar{X}_{\boldsymbol{n}}^{(j)}]\right|$$

$$\leq n_{1,2}\sum_{l=2}^{4}E[(\bar{X}_{\boldsymbol{n}}^{(l)})^2] + 2n_{1,2}\sum_{l=2}^{4}\sum_{j=l+1}^{4}\left(E[(\bar{X}_{\boldsymbol{n}}^{(l)})^2]E[(\bar{X}_{\boldsymbol{n}}^{(j)})^2]\right)^{1/2}$$

$$+2n_{1,2}\sum_{j=2}^{4}\left(E\left[\bar{X}_{\cdot\cdot}-\sum_{l=2}^{4}\bar{X}_{\boldsymbol{n}}^{(l)}\right]^2 E[\bar{X}_{\boldsymbol{n}}^{(j)}]^2\right)^{1/2}$$

$$\leq n_{1,2}\sum_{l=2}^{4}E[\bar{X}_{\boldsymbol{n}}^{(l)}]^2 + 2n_{1,2}\sum_{l=2}^{4}\sum_{j=l+1}^{4}\left(E[\bar{X}_{\boldsymbol{n}}^{(l)}]^2E[\bar{X}_{\boldsymbol{n}}^{(j)}]^2\right)^{1/2}$$

$$+8n_{1,2}\sum_{j=2}^{4}\left(\left(E[\bar{X}_{\cdot\cdot}^2] + \sum_{l=2}^{4}E[\bar{X}_{\boldsymbol{n}}^{(l)}]^2\right)E[\bar{X}_{\boldsymbol{n}}^{(j)}]^2\right)^{1/2} \to 0,$$

as $k_h/n_h \to 0$ and $k_h, n_h \to \infty,\ h=1,2.$ $\square$

Proof of Lemma 3. By the inequalities $|\Psi(z)| \le |z|^{2+\rho}$, if $|z|<1$, $0<\rho<1$, and $|e^z-1| \le |z|$,

$$|\Psi(z)| \le |z|^{2+\rho} I_{\{|z|<1\}} + \left(|e^z-1| + |z| + \frac{|z|^2}{2}\right) I_{\{|z|\ge 1\}} \le 4|z|^{2+\rho}. \tag{21}$$

Thus, by (21) and Inequality A,

$$\begin{aligned}
|R_{\boldsymbol{n}}| &\le \frac{4d_{1,2}}{b_{1,2}} \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \left| it\left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2} \sum_{\boldsymbol{i}\in\mathcal{B}_j} (X_{\boldsymbol{i}} - \bar{X}..)I_{\boldsymbol{i}} \right|^{2+\rho} \\
&\le \frac{d_{1,2}}{b_{1,2}} \left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2} \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \left(\left|\sum_{\boldsymbol{i}\in\mathcal{B}_j} X_{\boldsymbol{i}} I_{\boldsymbol{i}}\right|^{2+\rho} + \left(k_{1,2}|\bar{X}..|\right)^{2+\rho} \right) \\
&\le \frac{d_{1,2}}{b_{1,2}} \left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2} \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \left|\sum_{\boldsymbol{i}\in\mathcal{B}_j} X_{\boldsymbol{i}} I_{\boldsymbol{i}}\right|^{2+\rho} + d_{1,2}\left(\frac{16t^2 k_{1,2} b_{1,2} \bar{X}^2_{..}}{n_{1,2}d_{1,2}}\right)^{1+\rho/2} \\
&= R^{(1)}_{\boldsymbol{n}} + R^{(2)}_{\boldsymbol{n}}, \text{ say.}
\end{aligned}$$

If we choose ρ such that $0<\rho<\delta\wedge 1$, then by the Lyapunov inequality, $E[|X_{\boldsymbol{i}}|^{2+\rho}] \le (E[|X_{\boldsymbol{i}}|^{2+\delta}])^{(2+\rho)/(2+\delta)} \le \tau_\delta^{(2+\rho)/(2+\delta)}$. This implies that a modified version of Lemma 2(ii) can be used, i.e., we can use Lemma 2(ii) with δ and τ_δ replaced by ρ and $\tau_\delta^{(2+\rho)/(2+\delta)}$, respectively. Hence, if $k_h \ge m_h$, $h=1,2$,

$$E[R^{(1)}_{\boldsymbol{n}}] \le |16t|^{2+\rho}\tau_\delta^{(2+\rho)/(2+\delta)}\eta m_{1,2}^{1+\rho/2}(b_{1,2}/n_{1,2})^{1+\rho/2} d_{1,2}^{-\rho/2} \to 0, \tag{22}$$

as $n_1, n_2 \to \infty$, and if $n_h \ge m_h$, $h=1,2$,

$$E[R^{(2)}_{\boldsymbol{n}}] \le |16t|^{2+\rho}\tau_\delta^{(2+\rho)/(2+\delta)}\eta m_{1,2}^{1+\rho/2}(k_{1,2}b_{1,2}/n^2_{1,2})^{1+\rho/2} d_{1,2}^{-\rho/2} \to 0, \tag{23}$$

as $n_1, n_2 \to \infty$, which implies that $R_{\boldsymbol{n}} \xrightarrow{P} 0$ as $n_1, n_2 \to \infty$. $\square$

Proof of Lemma 4. Define $\tilde{\mu} = \hat{\mu} - \mu = \bar{X}..$, $\tilde{r}_{i_2} = \hat{r}_{i_2} - r_{i_2} = X_{\cdot i_2}/n_1 - \tilde{\mu}$, $i_2 = 1, ..., n_2$, and $\tilde{c}_{i_1} = \hat{c}_{i_1} - c_{i_1} = X_{i_1\cdot}/n_2 - \tilde{\mu}$, $i_1 = 1, ..., n_1$. By inequality (21) and Inequality A,

$$\begin{aligned}
|R'_{\boldsymbol{n}}| &\le \frac{d_{1,2}}{b_{1,2}} \left| \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \Psi\left(it\left(\frac{b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1/2} \sum_{\boldsymbol{i}\in\mathcal{B}_j} (X_{\boldsymbol{i}} - \tilde{\mu} - \tilde{r}_{i_2} - \tilde{c}_{i_1}) I_{\boldsymbol{i}} \right) \right|^{2+\rho} \\
&\le \frac{d_{1,2}}{b_{1,2}} \left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2} \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \left|\sum_{\boldsymbol{i}\in\mathcal{B}_j} X_{\boldsymbol{i}} I_{\boldsymbol{i}}\right|^{2+\rho} + d_{1,2}\left(\frac{16t^2 k_{1,2} b_{1,2} \tilde{\mu}^2}{n_{1,2}d_{1,2}}\right)^{1+\rho/2} \\
&\quad + \frac{d_{1,2}}{b_{1,2}} \left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2} \sum_{j\in\mathcal{J}_{\boldsymbol{n}}} \left|\sum_{\boldsymbol{i}\in\mathcal{B}_j} \tilde{r}_{i_2} I_{\boldsymbol{i}}\right|^{2+\rho}
\end{aligned}$$

$$
+\frac{d_{1,2}}{b_{1,2}}\left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2}\sum_{j\in\mathcal{J}_{\boldsymbol{n}}}\left|\sum_{\boldsymbol{i}\in\mathcal{B}_j}\tilde{c}_{i_1}I_{\boldsymbol{i}}\right|^{2+\rho}
$$

$$
= R_{\boldsymbol{n}}^{(1)} + R_{\boldsymbol{n}}^{(2)} + R_{\boldsymbol{n}}^{(3)} + R_{\boldsymbol{n}}^{(4)}, \text{ say.}
$$

From (22) and (23) we see that $R_{\boldsymbol{n}}^{(h)} \xrightarrow{P} 0$, $h=1,2$, as $n_1, n_2 \to \infty$. By Inequality A and Lemma 1 we have, for $i_1 = 1, ..., n_1$,

$$
E[|\tilde{r}_{i_2}|^{2+\delta}] = E\left[\left|\frac{X_{\cdot i_2}}{n_1} - \tilde{\mu}\right|^{2+\delta}\right] \le 2^{1+\delta}\left(E\left[\left|\frac{X_{\cdot i_2}}{n_1}\right|^{2+\delta}\right] + E[|\tilde{\mu}|^{2+\delta}]\right)
$$

$$
\le 2^{1+\delta}\left(\frac{\tau_\delta\eta(8m_1)^{1+\delta/2}}{n_1^{1+\delta/2}} + \frac{\tau_\delta\eta(64m_{1,2})^{1+\delta/2}}{n_{1,2}^{1+\delta/2}}\right) \le \frac{\tau_\delta\eta(256m_{1,2})^{1+\delta/2}}{n_1^{1+\delta/2}}, \quad (24)
$$

and thus, by the Lyapunov inequality, $E|\tilde{r}_{i_2}|^{2+\rho} \le (\tau_\delta\eta)^{(2+\rho)/(2+\delta)}(256m_{1,2}/n_1)^{1+\rho/2}$. By Inequality A and Lemma 2(i), with ρ instead of δ, $\tilde{r}_{i_2}$ instead of $X_{\boldsymbol{i}}$, and with $(\tau_\delta\eta)^{(2+\rho)/(2+\delta)}(256m_{1,2}/n_1)^{1+\rho/2}$ instead of τ_δ, we get (assuming $k_2 \ge m_2$),

$$
E[R_{\boldsymbol{n}}^{(3)}] \le \frac{d_{1,2}k_1^{1+\rho}}{b_{1,2}}\left(\frac{16t^2 b_{1,2}}{n_{1,2}k_{1,2}d_{1,2}}\right)^{1+\rho/2}\sum_{j\in\mathcal{J}_{\boldsymbol{n}}}\sum_{i_1=j_1}^{j_1+k_1-1}\left|\sum_{i_2=j_2}^{j_2+k_2-1}\tilde{r}_{i_2}I_{\boldsymbol{i}}\right|^{2+\rho}
$$

$$
\le (2^{15}t^2 m_2 m_{1,2})^{1+\rho/2}\eta(\tau_\delta\eta)^{(2+\rho)/(2+\delta)}(b_{1,2}k_1/(n_{1,2}n_1))^{1+\rho/2}d_{1,2}^{-\rho/2} \to 0, \text{ as } n_1, n_2 \to \infty,
$$

which implies that $R_{\boldsymbol{n}}^{(3)} \xrightarrow{P} 0$ as $n_1, n_2 \to \infty$. Likewise, $R_{\boldsymbol{n}}^{(4)} \xrightarrow{P} 0$, and so the desired result follows. □

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